MATHEMATICS TEACHING

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MATHEMATICS TEACHING

Contributions to the journal should be sent to the Executive Editor. Contributors are asked to keep mathematical notation as typographically simple as they can, and to submit illustrations, where necessary, in a form suitable for reproduction. Enquiries about distribution should be addressed to the Secretary.

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This Bulletin appears three times yearly.

A MESSAGE FROM THE PRESIDENT

I was extremely pleased when I heard last year that the Association was still keeping me in its midst by making me its first President after I had been the first Chairman of its Committee for about 5 years.

When in the Committee we learnt to work together as a team, the obvious thing was to know that we had to learn a great deal and that the best way for us to grow was to go to the teachers and the students where they are. Our humility and our enthusiasm gave us the chance of appearing what we were: keen students of the science of teaching mathematics. Because we learnt by observation, by studying pupils' mistakes and tracing them to the mental structures and the techniques we used, we struck the right direction and all over the country first, in several other countries afterwards, our Association became known as formed of a group of earnest workers; practical people who had ideas and a great love for truth. All this is shown by the fact that we are over 900, although this is only the fifth A.G.M.

When I went to Canada last Summer I found that teachers there wanted to share our experiences and thirty gave me their subscription for 1957. Our membership is international because our problems are simply human. Here in Addis Abeba I am learning every day things that are important for you, too. What you taught me, in our contacts, serves me to be more efficient in this country. There is so much for us to discover in our activity and so much improvement to bring to our work in order that the joys that we receive from our success shall be the everyday feature of our function, that *Hope* is the sign under which we can now work. Indeed, when we think of the smiles and bright eyes of so many who formerly were condemned to remain untouched (or even disgusted) by mathematics and now enjoy and even love it, we can see our work as providing spring-boards for the young generation, and no longer strait-jackets.

In hope, I greet you on this occasion of our A.G.M. wishing for you as much happiness in your teaching as I know you could get if you met your classes as sources of the most important knowledge you need in order to be successful: if you meet your activity as a continuous creation of more understanding, more insight into mysteries and challenges.

As we are at this time only beginning to take stock, let us not forget that there are always new openings after the next ones.

C. GATTEGNO,
ADDIS ABEBA. 9/2/58.

THOUGHTS ABOUT PROBLEMS

M. GOUTARD

(translated from French)

The kind of problems that we are in the habit of inventing for children to solve are, it seems to me, concerned with very precisely determined situations: in them a number of terms are brought together in a certain relationship which is very set and fixed. From this static scheme there is the one possibility only—that of finding a single missing term. Actually, any one of the terms may be omitted, in succession, and in each case the child is asked only to fill in the blank. We could compare such problems, and the words in which they are stated, to a jig-saw puzzle which the child must reconstruct in order to specify which piece is missing. Or, to change the metaphor, he is given the ends of a chain and must assume that the other links exist.

Unless an analysis of the mathematics involved, and of the dynamism of the relationships, has been learnt previously, the child finds that he cannot solve these problems: this is made worse because the facts are buried in a concrete form, which adds to the confusion by making it difficult for the child to find the relationships, since they are not visible as such. This explains the successful results which can be obtained by a teaching method that Cuisenaire rods make possible, for a mastery of dynamic schemes is gained which can be applied to any concrete situation. The child is at ease in using his knowledge, and *immediately* finds the answers to the problems put to him.

From this one could say that the question of problems is solved. I ask myself, however, whether in such a case one is justified in speaking of "problems". Rather, it seems to me that we have arrived at a point at which the problem has disappeared: at this level there is no challenge to the mind which deserves to be called a problem.

What strikes me is that whenever we set a problem which is exactly determined (which it is, if only one solution is possible) we imply a mind capable of its *instant* solution. In proposing a problem to a child, we are assuming that he is capable of interpreting it to himself. But as soon as a problem is correctly posed in the mind, it is both understood and solved by the same act of thought. All that is required is a little time in which to formulate the reply. But this time is not being used creatively. It is occupied only in shifting from one point to another by means of acquired mechanisms. Just as one cannot instantaneously transfer oneself to the other side of the road, so does it take the mind a little time to perform certain operations. But the problem itself was solved as soon as it was put and seen. And it is a well known fact that, in the history of science, an unsolved problem is one that is wrongly posed and that the person to solve it will be the one who asks himself the right questions.

In putting any particular question to a child we assume that, if he is able to put it to himself, his mind will work at just this level of reality. If this is so, the solution will be found immediately, in which case there is no more taking place than the exploration of a certain horizontal level. We are giving training (certainly useful) in some dexterity, but are not concerned with true creativeness. If, on the other hand, the child does not move on this level, the problem cannot be expressed

in his mind: he must first make the preliminary analysis referred to above, and not until he has done this will he be able to rise to the appropriate level.

We seem to be faced with these alternatives:

1. We may pose to a child problems which he can solve only by rising to a higher level through "inventing" the mathematics he needs. But, besides the dangers of affective blockage that such a method carries, it is also suspect in that it does not give the opportunity for a sufficiently free analysis of the mathematics involved, since it provides only one static example of a dynamic scheme; a necessarily constrained aspect of what is merely a contingent state.

2. We can make the algebraic analysis separately and in the abstract. From fruitful awarenesses, the child acquires such ease and freedom that thereafter he will have no difficulty in applying his knowledge to problems. But now that he is only applying his knowledge, his mind is no longer being used creatively. He has reached the point at which the "problem" has disappeared.

Consequently, it seems to me that in order to find a way out of this alternative we must form a different conception of what a problem should be.

Let us bear in mind that what appeared to be the most important thing was not the solution of a problem but the way of posing it. Instead of requiring that the child shall solve problems, mathematical education ought to stimulate him to ask himself questions (clearly more is envisaged than the writing out in his own words of little stories on a familiar theme: instead we are contemplating a reversal of what is usually done). The need to give children a sense of problem arises because we need creative mathematicians and not only people who know mathematics and can express clearly what they know. Suppose that, instead of having them set for him, the child himself meets, invents and constructs his own problems! For what matters is the questioning; reflection is only a question in action. Questioning should come not from the object but from the subject. It is man who asks questions of nature; there is a response if the questions are well framed. But in our classrooms what happens is just the contrary: objective reality is used to put questions to the child, and the questions are put in such a way as to leave the possibility of only one answer.

I want to show that a problem taken at the "object" level, about a reality already fixed, cannot be other than artificial since the facts given already include the solution. The most banal example will prove this. "Some goods cost 100 francs. They are sold for 120 francs, what is the profit?" When reality questions in such a way, it does it artificially; it is pretending to ignore what is already known, since it is obvious that the profit is fixed *at the same time* as the selling price. If there was a problem at all, it occurred before this. What may have been the possibility of a problem lies in the "fact" that the goods costing 100 francs had to be sold at a certain price: one might here have paused to consider if it would have been better to sacrifice a little profit in order to assure a larger sale. But it is made clear that as soon as one term is fixed so is the other also, and that thus there is no problem. In this we also see that the "time" of the resolution of such a classic problem is only an arbitrary and artificial time, which has no bearing on the time which relates to the solving of the real problem as it is experienced by the solver.

We guess now how it will be possible to find an issue to the alternative above. Instead of starting with problems situated at an objective level in a reality completely determined, we must start with problems situated at the level of the subject in a reality not completely determined. The problem will consist in determining it and in turning it into a "solved reality". The time factor entering into the solution will be a creative time used mentally and actually: having a proper function and not (as in the trivial example) just as an artifice.

It is obvious that such a way of looking at a problem accords with true mental activity. In life we are faced with problems which are at first more or less undetermined. People live at different levels of awareness of what reality is, which gives an illusion that another person could enter a situation that is from the first perfectly determined. To his awareness it is at first incompletely determined; it is through the effort to increase this awareness that he determines it. Initially there is nothing but a "fact": e.g. the necessity of constructing a house; or the presence of a certain number of objects; or the apparent movements of the sun, etc. . . . These facts offer a base for the progressive development of all that is possible.

In all fields of education, and especially with young children, we must start with indefinite situations (for such is the reality in which they live). If in such a situation it is the size or number of objects that must be considered, the problem becomes mathematics. With the little ones who are just beginning to count, I suggest that problems arise from facts as simple as this: someone has given us a certain number of things (pictures shall we say). What shall we do with them? Each child starts off in any way he likes, to find out what can be done from the fact that he has this number of pictures. All kinds of problems arise spontaneously. The harvest cannot fail to be rich. From this lively beginning with its possibilities of flying off in all directions, however, certain especially interesting avenues can be found and explored: for example, collecting sets of cards given out in succession. This provides an endless series of possibilities obtained in a precise manner. In another direction, on the contrary, it can happen that the possibilities become rarer and rarer, that we tread ways more and more restricted and impose on ourselves conditions more and more difficult, until we reach the point at which the large or small number of terms left are in a state of completely static equilibrium. To take a simple example, let us suppose that we decide to take two sets of cards: if we fix only the difference between the number of cards in the two sets (say by giving three more cards to one child than to another), there is an infinite number of possibilities; if only the sum is fixed (the cards of one set being shared between two children) the possibilities form a definite series; if both sum and difference are fixed, then only one solution is possible.

So, placed in an indefinite situation he wants to determine, the child uses his imagination towards an aim that he will reach through action. In the process he finds the data of his problems. According as he wishes to go in one or another direction, he sees which terms he needs to support his actions and adjust his operations. But the terms relative to the ones he uses are implied by his choices, and he becomes aware both of the terms and of the relations. His deliberate constructive action gives him the power to analyse a reality which at first was only potential. Having introduced more and more "necessities" until he has reached a certain point

in the field of determination, he can go back, unlinking the related terms and freeing the possibilities, until there remains only one thing given, the number of possibilities becoming again infinite—the return gives a rational development of all possibilities.

The traditional problem is found at the final stage of the first movement in this scheme: the one which goes from relative indetermination into complete determination. To bombard the child with disguised versions of some fixed types of problems is quite unnecessary, whereas from any real fact an infinite number of problems can be developed.

It is worth saying that if we start from what is indefinite the child ceases to be obsessed by *the* answer, a magic figure which he must find at all costs. Since he is no longer confined in a maze from which he must find the one exit, any issue is valuable when he can justify it. The child's mind is concentrated on his own aims and actions. Each child can respond at his own level, giving his own solutions and explaining them freely. Also, the data which he finds and brings in are those which suit the growing demands for precision corresponding to his own awareness, and hence he has full understanding of their function. He no longer finds himself embarrassed with more terms than he can deal with, having no idea why they are included.

Perhaps all this will throw some light on problems about unequal parts, as in mixtures, alloys, etc. These are supposed to be difficult and many methods are used to solve them, e.g. graphs, reasoning, mnemonics, although it is considered preferable to use alegebra in a mechanical way once the formation of equations has been learnt. I quote a typical example from a book intended for teaching in the fourth year of a secondary school: "Two casks contain equal quantities of wine. The wine in the first cask costs 70 francs per litre, and that in the second costs 75 francs per litre. Half the contents of the second cask are poured into the first one, and then half of this mixture is poured back into the second cask. The contents of the latter then have a total value of 730 francs. Find what quantity of wine each cask held initially." The way the statement is made almost leads one to believe that the mixture has been made at random and as it happens one becomes aware of its price, then wondering what the composition could be. Certainly some children do understand it like that. As if the composition had not been known first, in order for the price to be evaluated! Since this is the case, the real problem requires that we shall find the proportions in which the elements must be combined so that a predetermined price can be charged. So I want to show that it is only when children are given the opportunity to "live" the whole problem just as it naturally appears to anyone who actually wants to make mixtures of any kind, that they will be able to understand and master the compensating dynamics involved in such problems.

What we want, when we make mixtures, is to obtain intermediate quantities. As it is easier to realise what happens when the elements preserve their identity within the mixture, a good beginning could come from imagining oneself to be a carpenter who has to make a floor with blocks of wood of different lengths. First we decide on the total length (and discover how the elements can be combined in various ways to make up this length), next we consider the number or size of the elements (trying to find out how to combine and arrange what we have, in order to make another new length each time). In each case we examine the field of

possibilities, consider new demands and make a choice that will fit the concrete situation. The analysis takes into account the factors involved and the part they play: the greater the variety of elements, or the smaller the elements, or the smaller the differences between them, the greater is the number of possible lengths that can be formed. Then if we become interested in the work of chemists or of salesmen who want to mix liquids of varying price or consistency, we can start off by finding the composition, or the price, of mixtures which can be made, and discover the limits of variation that we can choose from. Finally we may investigate how to obtain a mixture having such and such a consistency or price that we presume to be possible. It is soon seen that if the number chosen is the arithmetic mean of the given elements, the mixture can easily be made. Other cases may be more difficult. Suppose we start with liquids of value $3u$ and $10u$ and decide to make a mixture from these that has any of the intermediate values we happen to choose. Cuisenaire rods will provide a particularly useful aid in the work. A child decides to make a mixture of value $6u$. He begins by mixing one litre of each, so that he gets $10u+3u=13u$ (an orange and a light-green rod), when the liquid he wants should be only $2\times 6u=12u$ (2 dark-greens), so he finds that he has too much. If he adds a litre of the lower rate, then for three litres of the mixture, $13u+3u=16u$ (an orange and 2 light-greens), although he should obtain $3\times 6u=18u$ (3 dark-greens). Now he has not enough, so decides to add a litre of the higher value. He continues in this way until he reaches the exact mixture he wants (such that the line of orange and light-green rods is the same length as the line of dark-green rods). This happens after mixing seven litres: four of value $3u$ and three of value $10u$ (4 light-green rods and 3 orange rods = 7 dark-green rods).

When we examine all the solutions obtained by the various members of the class, we see that each of the different mixtures chosen was found by combining seven litres, but in different proportions. This comparison confirms the awareness that what concerns us is the difference between the two quantities, and as a result the pupil can appreciate the way in which intermediate quantities could be formed. With seven litres of value $3u$ (a train of 7 light-green rods) and seven litres of value $10u$ (a train of 7 orange rods) we can make all possible mixtures. By interchanging litres of $3u$ and $10u$ (which is to replace a light-green rod by an orange rod and vice versa) our first mixtures have values $4u$ and $9u$ per litre. Explaining in terms of the rods: to use an orange instead of a light-green adds "7" to the total length which initially was of 7 light-greens. This is equivalent to supplying one white rod to each or to exchanging light-green rods for crimson rods. Similarly, to remove one of the orange rods and put a light-green one in its place can be considered, alternatively, as changing to a train of blue rods. At the next exchange of a light-green rod and an orange one, the two new mixtures, 5 light-green + 2 orange and 2 light-greens + 5 orange, are equivalent to trains of 7 yellow and 7 brown (and have values $5u$ and $8u$ per litre). We continue in this way until we have made the total of all possibilities (between 3 and 10). The development gives insight into the whole scheme so that a child will see at once that if he wants a mixture of value $8u$, say, he combines the liquids $3u$ and $10u$ in the proportions of 2/7 and 5/7.

On returning to the former problem that was supposed to be difficult, we find that the children have become experts on mixtures of all kinds so that they find

is already resolved (as, truly, it is) without their even having had to pick up a pencil. From the values to be combined, of 70 fr. and 75 fr., the first thing we see is that the result lies between these, and the second obvious fact is that it will be a multiple of 5 if people dealt with integers. Seeing the figure 730 fr. suggests at once that we are concerned with 10 litres at 73 fr. per litre, composed of 2/5 wine at 70 fr. (4 litres) and 3/5 wine at 75 fr. (6 litres). All that remains to be done is to take into account that the second cask contains half of the 70 fr. wine, which tells us immediately that both casks originally held 8 litres.

Problems of this kind are often stated with one extra fact that is irrelevant (which may be the total number of litres of the mixture), while a direct insight makes for great economy of labour. There is now no laborious forming of equations followed by blind mechanical resolution; no tedious arguments based on false assumptions; no imaginary substitutions which are so long winded and involve repeating the same process over and over again. "Reasoning" is brought in because one does not "see". But the child who can see is only the one who is allowed to experience the true problem in all its vicissitudes and not its ***a priori*** resolution: that is, the child who mixes and who provides himself with the changing reality on which he works so that he can observe the effects of what he does to it, creating various and interesting fields of determination. The highly structured "traditional" problem should be used only as a final test by which a child proves to himself that his mastery of mixture problems is such that he can also "see" when the problem is reversed. This reversibility is now possible and time does not enter any more, since the whole problem has been exhausted through true problems clearly seen.

My thoughts now turn in another direction. I mentioned earlier that all teaching must begin with situations that are not precisely determined, but which need to be formed, and that by taking a certain point of view on it, the problem becomes mathematics. I now want to discuss this *instant* at which the problem changes into mathematics.

If a problem is concerned with a concrete reality, this might be of such a quality that it could not at once be dealt with as mathematics without imposing arbitrarily on the child. Telling the time is a good example. It is not a foregone conclusion that a child is ready to learn to tell the time simply by virtue of the fact that he knows how to multiply by 5. There is the risk of teaching nothing but a meaningless ritual. I will describe an experience I had with five-year-olds that shows at what moment such knowledge can be introduced.

We were in a new classroom which had an electric clock; in the children's stories this clock often entered as a mythical person of evil repute. I then became aware that the children did not realise that the clock hands moved uniformly, and that for them it lived between stopped moments. They thought the hands made a small movement, stopped for a longer period, moved on a little, stopped again, and so on. "But," I asked, "what about when it is completely stopped?" One sensitive and intelligent child gave an answer to which the other children assented: "When we turn round and round we get giddy and then have to stop. It is the same with the clock, for although the hands don't go round so quickly, the force of turning makes it giddy so that it has to stop to recover." Interest being aroused we watched carefully to see when the hands moved. To pass the time the children began to count. Each time the hand moved when they had reached a different number, since

their speed of counting was not regular and they also lost count in bursts of laughter when they saw the hands move. They noticed the variability, but attributed it to the moodiness of the clock or to its degree of giddiness. This continued for some time, until one child said firmly: "No, the clock's hands move evenly, it is we who count too quickly." What a turning of the tables! From measuring its movement, we had discovered instead that it measured our own! We sat at the table with our eyes glued to the clock and beat out the measure, determined to make it as regular as we possibly could: now we found the same result every time to within one count. This sudden deep awareness of the relation of the subject to his world led to endless questions about clockmaking, etc. and to further widening of the field of consciousness of time (what would happen if clocks did not mark the passing of time, first thinking of the disturbance that it would cause in our little schoolchildren's world and then going on to imagine the chaos that would enter the whole of social life). It showed me that it is not until children are conscious of problems of time, and of the correspondence of the different movements in the world, that they are ready to consider a mathematical point of view.

Further reflections on this revealing example make one wonder if the difficulties which some adolescents have in applying mathematics to practical problems are not caused by their having been plunged too soon into quantitative aspects of it. A qualitative study should first clear the ground, sweep away obstacles and provide the opportunity to develop awarenesses that are essential to a proper understanding of the practical situation involved.

To summarise, what we are recommending is an existential conception of problem: that problems should form and mould themselves from the spontaneous contours of the child's thought, favouring its movement towards greater maturity. It is no longer a question of puzzles perfectly complete, and independent of the subject who has to deal with them. The individual should be called upon to interfere effectively and efficaciously in order to raise reality to a higher level of structure. This is why a true problem of consciousness must be a problem having true duration, requiring time for its solution: which means that the mind is alert and, from its growing awareness, is constructing its world.

PUZZLING TRIANGLES

The triangle with sides 3, 4, 5 is an old friend. The property which interests us here that it is a triangle whose sides are consecutive whole numbers and whose area is a whole number too. The next triangle with this property has sides 13, 14, 15. What is the next; and what is the general rule?

(*Solution on page* 48).

MATHEMATICS WITH BACKWARD BOYS - AN APPROACH

S. HODGKINSON

These boys will become navvies, or, probably at best, lorry drivers or any job requiring little skill in between these two extremes. As one of my boys said recently, when asked by Y.E.O. what job he wanted, "Somefink wiv no finkin' in it." Granted this is an extreme case, but it happened! He was by no means the least intelligent!

What mathematics will such boys *need*? It is necessary first to know the boys and their background and future—the above gives some indication of that.

The basic minimum needs appear to be the ability to do plain counting probably not beyond 100, and to be familiar with multiplication tables so that shopping and household problems such as building a shed, or papering a room, can be dealt with.

But much even of the above is quite unnecessary. One can stop after the words "shopping problems".

I had this borne in on me very forcibly when one day I set this problem to a group of my weak boys.

"Your wages are £5 10s. a week. You are to receive a rise of 10 per cent. What will your new wages be?" If I remember rightly, one boy out of 15 had the correct answer!

Thinking I was on to a good thing I pointed out that surely it was worth a little effort to master this type of calculation for such a situation could easily arise at any time in the very near future for themselves. This only provoked derisive laughter (as much as they dared). That, they said, presented no problem at all. In every circle there was always one chap "good at figures" who would do the problem for them! Therefore why should they be bothered! I felt very impressed by this (and depressed!) It is undoubtedly the key to much trouble in school with lazy and weak school leavers. There is no *REASON* for working mentally.

To the Schoolmaster and Social Worker generally, this is by no means the end of the problem. If it is agreed that so far as prime necessity goes, the ability to spend one's money without being cheated is all that is necessary, the great fact remains that all, *ALL* will shortly be fully pledged voting citizens in a Democracy.

Here, surely, is a challenge and one which cannot and should not, be ignored. Yet a frontal attack on the problem is quite useless, in fact I have yet to find how to attack it, though I am now certain about the direct, all-out frontal attack. That is doomed to utter failure.

These boys are lazy-minded and will not make use of their intellectual equipment unless

(a) forced by circumstances
(b) convinced of the real value to themselves.

Sir Richard Livingstone, whose two little books *Education for a World Adrift* and *The Future in Education* contain all the general principles for education necessary, puts his finger on the spot when he says, quoting Aristotle, "The young are not fit to be students of politics, for they have no experience of life and conduct,

and it is these that supply the premises and subject-matter of this branch of thought."

This general principle is, I am sure, of general application, and here, when we are trying to convince boys of 14-15 of the necessity for understanding certain matters of mathematics because it is necessary to them as future citizens, voters and good neighbours, they will not respond because, apart from weakness intellectually they have no experience of the wider life.

A very recent experience with a 15-year old backward group brought this home to me, for a reason which will appear later. I was discussing the census with them, explaining what it was, how often it happened and the kind of figures it produced for this country. After a few minutes one boy asked "Why should we want to know how many people there are in the country? What is it all for?" The question "What for?" is one always uppermost with these boys.

Now the idea of the necessity of a periodic census is so familiar and unquestioned among educated citizens that I must confess I had not previously prepared an answer to this and had hurriedly to search around for a convincing one, remembering that some solid, probably financial, gain must be shown to result! Just at that time the Labour Party was publishing details of its proposed "pensions for all" plan. I hit upon this, pointing out that for any group of persons wishing to work out such a scheme, the cost was a very essential item, and how could the cost be worked out if the numbers of people in the various population categories (not these words, of course) were not known? This satisfied my enquirer entirely. I had shown that money would result to some people from this operation and that was good enough! But I am not always so lucky.

Some Long-term Experiments.

The foregoing gives an estimate of the problem.

What follows is no solution, but is an account of a number of pieces of experimental work of a long term nature in each case, all of which have arisen quite naturally as I have faced the problem in its varying aspects.

I. I took over my present headship in November, 1950. At Christmas, 1950 it became necessary to make certain small time-table changes. In doing this I awarded myself the task of taking the mathematics of the most backward Fourth form boys until they all left school in July. They had one lesson every day from 9.20 till 10, under that regime. For these lessons I made myself entirely responsible for the next two terms. I have detailed notes still in my possession of everything in connection with this work.

The plan I adopted was a very simple one, and was one which, in those circumstances I had never adopted before, and have never adopted since. I determined to teach these boys the elements of arithmetic. Four Rules with ordinary numbers, money, fractions (no denominator higher than 16), decimals to two places and the simplest applications to everyday life. During the first week I gave the boys a test covering the above topics but without applications.

There followed each week, four lessons on a particular type of sum with a Friday test on that type of work together with revision questions. *Each Friday, good progress appeared to have been made.*

During the last week after six months real concentrated effort, and, for them, the boys worked well, I gave the boys exactly the same test, with the following results (ignoring a considerable number of boys who had left at Easter and a few others who were absent) :—

Out of *19 boys* who took the test on both occasions, the scores of 13 of them either remained the same or improved by only one point in 18 (the maximum mark) or actually were lower than 6 months previously,

6 improved more than 1 point, *but only one showed really significant improvement*. In effect, after six months' hard work, no gains were recorded.

I have quoted this on many occasions and the remark made by a student seems to be apposite, "Clearly they had reached their ceiling!"

One further point should be mentioned. Many teachers would probably feel that a basic weakness (more than that) in tables might be behind these results. This was countered by a decision I made at the outset, that at this stage it was hopeless to expect to teach tables, so every boy prepared a "Table Square" which he was allowed to use always, including during his tests.

II. A year or so later I divided the mathematics teaching of my senior forms into, first, Arithmetic, taken daily by the Form Master and, second, a period of approximately one hour weekly of General Mathematics, taken by myself. The domestic reasons for this I will not go into here except to say that in ideal circumstances I do not myself agree with this division.

The work I did with the backward Fourth form consisted largely of practical problems based on outdoor measurement such as heights and distances and the scale drawings which followed. Certain difficulties immediately became apparent.

(1) It is *very*, very desirable to base this work on REAL measurements. Such measurements are rarely convenient for simple calculations or drawing, yet it is necessary that at the end the objective shall be secured.

(2) This shows itself first in the selection of a scale. A distance of 93 ft. has been measured. This now has to be drawn on paper of a certain size and a certain scale has to be chosen, with or without help. 10 ft. to the inch immediately suggests itself but is ruled out by the size of normal exercise book paper, 8 in. x 6 in. One has to use 20 ft. to the inch which presents insuperable difficulties to these poor lads! Those who remember my letter to the Times Educational Supplement of December 6th last will remember that even 10 ft. to the inch is too difficult in some cases!

(Lately I have found 1 mm. to 1 ft. has come in very handy!)

(3) The final conversion of the height on the drawing to the actual height gives much trouble in these *real* cases because it hardly ever produces an exact result.

In the course of this series of lessons covering one year I also noted that very many boys had absolutely no idea of judging a distance, e.g. the length of their form room wall in yards, or indeed in some extreme cases, the length of one yard itself.

I felt however, at the close of this course and in spite of much frustration and difficulty, that this was nearer the type of work these boys needed than more and more "sums".

III. The following year I extended the division of mathematics teaching as described above to Third forms, still taking all the General Maths myself. This meant that I had these backward boys for one hour per week for a maximum of two whole years.

I spent the greater part of the time during the third year (penultimate leaving year) on the very simplest problems in Air Navigation, the teaching of which to A.T.C. classes I had had experience during the war.

I should like to interpolate a remark "in passing". One of my aims became to give a boy the ability to do something of which he can be proud. A neat, carefully drawn and accurate scale drawing *is* such a thing, the amount of mathematics yet being negligible.

A very brief summary of this work follows:

(1) Drawing triangles given usual data (I found many boys very weak on both *drawing* and measuring straight lines; they appeared in many cases to draw them with their feet and measure them by guesswork!)

(2) Direction in navigation—Angles from 0° to 360°

(3) The representation of "a wind of 025°/40 knots"

(4) The terms *Course, True Air Speed, Track and Ground Speed*

(5) The solution of problems of this general nature: An aircraft on a Course of 090° T of which the True Air Speed is 250 knots meets a wind of 025°/25 knots. What is its Drift, Track and Ground speed?

Interpolating on general teaching problems here, I would say that I always write up in a book, during the previous week, my teaching plan for the following week, leaving a column for comments. I have found the comments on re-reading them, most illuminating. After a lesson on wind direction I noted that "About half the form are not using protractors at all, merely guessing at the angles, or, rather, drawing something like the last drawing they saw on the blackboard. The next thing is to get this right."

The boys showed great interest in this in general, the interest only waning when, even here, they found it necessary to work *and think!*

IV. During the previous year we enjoyed a general inspection and, while being generally approving of our mathematics teaching, the inspectors disapproved, as I did myself, of the division into arithmetic and general mathematics. I decided to alter this for the following year and as a result took all the mathematics of the three Fourth forms myself. With the backward form I massed four periods into one half day so that I could take the boys out for the complete period on to the Wanstead Flats, about a mile distant for a simple course in surveying. The one remaining period was devoted to mental arithmetic on daily life problems.

It so happened that this particular backward Fourth form was composed of boys of a rather more agreeable nature than is usual, and the surveying course, consisting mainly of the simple traverse, went very smoothly. There are practical difficulties here, as usual, not the least is the weather. I find that it is very rare to be able to get much outdoor work done in January, February or March, so it was necessary to concentrate on the actual outdoor measurements in the Autumn Term leaving the drawing and calculation to be done in the Spring Term during bad weather.

This has the unfortunate result of

(a) boring the boys with routine measurements

(b) achieving no immediate connection between their field-work results and the final finished map or plan.

The work was done with a simple workshop-made horizontal angle measurer mounted on a tripod and distances were measured by pacing. I have never ceased to marvel at the astonishingly accurate maps which can be made by this method. We mapped the whole of the "Flats" by this method, showing all the main paths, clumps of trees, ponds, allotments and prefabricated housing areas. Finally, after all was complete (and not before), I bought a 25 in. to the mile Ordnance Survey Map and our own map was really remarkable when compared with this and considering the operators and their tools!

A few years later I repeated this experiment, but using a smaller area and mapping it in greater detail. The boys were a less agreeable crowd and lost interest in the actual measurement after a fortnight and had to be driven along. They produced some quite good drawings however, though only with much help.

During the Summer Term, and with a smaller group, many having left school at Christmas and Easter, further work has been possible with the plane table.

V. One morning as I was preparing to start my surveying class on their morning's drawing, one of the more garrulous among them (and one of the weakest intellectually) thrust in front of me a copy of "Punch" open at a whole page cartoon under which was the caption "247,632 persons were killed or injured on the roads of Britain last year". I can only vouch for the order of this number, but it certainly was over 200,000.

The boy then commented "but that can't be right, can it, sir, because there aren't as many people as that in the whole country!"

This gave me an idea. I referred to it earlier in the anecdote on the Census. Clearly the boy had no grasp whatever of large numbers. Anything over 100, or perhaps 1,000 was just "a very big number" and that was all. Yet to understand many newspaper paragraphs on quite ordinary topics, such as armies and navies, tonnages of ships, attendances at cinemas, audiences for TV programmes, etc., etc. it is *necessary* to know the meaning of very large numbers.

I therefore devised a series of simple lessons on significant figures leading to graphs of statistics, using only common words inside the boys' experience. I did not for instance use the word "significance" but substituted "importance".

For example, Unemployment figures between 1921 and 1931 were tabulated and reduced to understandable proportions by giving them to "the two most important figures" and then plotting them on a graph.

In this way it did prove possible to make these large numbers mean something.

VI. During this current year I am developing further the use of our own school playground (an ordinary asphalt patch), for a series of exercises in practical measurement, scale—drawing and surveying which can be dealt with in a one-hour period and which will not require us to take the mile long walk to the Wanstead Flats.

Subject to the strictures which have gone before, I think this, when developed into a routine technique offers the best chance so far of finding useful mathematical employment for these boys.

The exercises involve the following:

To draw a plan and to calculate the area from

1. Chain survey
2. Triangulation—measuring the three sides in every case, using "rays" from one angular point
3. Triangulation, using rays from one angular point, but measuring the rays only and the angles between them
4. Triangulation, using rays from a central point
5. A simple traverse
6. A plane table traverse
7. Normal plane tabling.

Of course the above involves

1. Measurement with ten-yard rope knotted each yard
2. Use of horizontal angle measurer
3. Scale drawing (The scale 1 mm. to 1 ft. has been found to be very suitable to ordinary Exercise book use).
4. The area of a triangle
5. The area of a trapezium
6. The use of the plane table.

Finally, one chief idea of all this was to give a clear idea of the size of an acre. We found that our playground was just over one third of an acre; so now the boys know that they will need to think of an area about 3 times the size of their playground when picturing an acre.

It was in the course of following this programme that I came across the appalling errors common with these boys which prompted me to write to the Times Educational Supplement.

On the very last occasion before writing this article, on which I took the class, I listed the chief errors made that morning. They were as follows:

Boys had been told to write

Area of triangle = (Base × Height) ÷ 2

at the head of each calculation.

It was pointed out to them that ÷ 2 should be included in each subsequent line of the calculation until the division was actually accomplished.

(1) This lead to its being included in *every* line including those *after* the division had been done. One boy divided by 2 twice as a result!

(2) 143 × 50. Many could not even start to deal with this. The "0" finished them

(3) 143 × 49. Some could deal with this successfully, but many worked 143 × 94 unwittingly!

(4) 2145 ÷ 2. Many achieved the answer 172 instead of 1,072, for obvious reasons.

In conclusion, there is here a dilemma. Anyone who has the solution has solved the problem of the mathematical education of backward boys.

I state it as follows:

"To keep backward boys interested, the subject matter of the course must be constantly changed, *but*, if anything is to be known thoroughly and especially by *backward* boys, it must be constantly practised."

So far I have not found a practical way of making these two essentials co-exist.

THE NEW GAMESMANSHIP

T. J. FLETCHER

It is rare these days to be able to explain the ideas involved in a new branch of mathematics in simple terms, and so a teacher can approach the Theory of Games with double pleasure. In the first place there is the delight of encountering a new and exciting branch of mathematics for the first time, and then there is the pleasure of realising that the essentials may be explained by means of simple examples which call only for the most elementary algebra.

For centuries the mathematical theories underlying many particular games have been known, but recently general techniques have been developed which are applicable to many apparently diverse games and also to other situations which have not previously been termed "games" at all. The early work on probability arose from the study of gambling, and later provided the theoretical background to statistics and insurance practice; in a rather similar way the present theories have arisen from another aspect of games (the outcome of choices) and are developing into a mathematical system which will apply to strategy and business organisation. This comparison can be pressed further because the policies to which the modern theory of games leads are policies of insurance against unnecessary loss. Mathematics encourages the virtue of prudence, and the only assistance it gives to the speculative gambler is to make clear to him the risks which he is running.

The founder of this new subject was John von Neumann, who died in February 1957 in Washington. A paper which he published in 1928 contained a proof of the fundamental theorem that the type of game which we are about to discuss always possesses a solution—that is, there is a definite course of action which it is best for the players to adopt. In 1947 von Neumann and Morgenstern published their book *The Theory of Games and Economic Behaviour*[1] and the subject was then well and truly started. This book is difficult and can only be recommended to specialists, but there are easier expositions such as the book by McKinsey[2], and, in some ways the most remarkable, *The Complete Strategyst*[3] by J. D. Williams, which attempts with astounding success to explain the basic principles by simple arithmetic without recourse to even the most elementary algebra. Most of the examples in this article are taken from these sources.

A theory of "games" may sound trivial, but "game" is used as a technical term which includes economic action, war, and almost any activity in which there is a conflict of interest and the possibility of choice. Professor R. B. Braithwaite gave his inaugural address on the theory of games and moral philosophy[4].

It is time to get down to some simple examples.

The first game we will consider is *Matching Pennies*. In this game my opponent and I place two pennies down on the table simultaneously; we can each decide whether to place the penny down with head or tail uppermost but of course we have no control over what our opponent does. The rules are simple. If both pennies show the same I have them, and if they differ then he has them. How should we plan to play the game? Ordinary common sense suggests that one should put down heads and tails in a random manner and hope, in the long run, to break even. If I favour heads the danger is that my opponent will come

to see that I am doing so and be wise enough to favour tails himself. If this happens I shall tend to lose. Therefore the policy of randomising heads and tails is an insurance against loss. This analysis is entirely commonsense, but it displays many general features of the theory. If we take a more complicated example we shall have to introduce numerical reasoning.

What happens if the rules are modified so that I gain or lose twice as much if I play heads as if I play tails? The situation may be described by a *pay-off matrix;* this is simply a table showing what happens in the different cases.

		B plays *H*	*B* plays *T*
A plays	*H*	2	−2
	T	−1	1

We will adopt the following conventions throughout. The game is between two players *A* and *B*. I will identify myself with the first player and indicate the different choices I can make by the rows of the table, the different choices which *B* can make will be indicated by the columns. The individual entries in the table show the outcome of the combinations of choices—the outcome as seen from my point of view. Thus if I play *H* and he plays *T* the corresponding entry −2 in row one and column two shows that I lose twopence.

To play the game I shall adopt a *strategy*—that is I shall play *H* and *T* at random but in a certain ratio a: b which I shall calculate. My opponent will also adopt a strategy and play *H* and *T* in the ratio p: q. (It is convenient to choose units so that $a+b=p+q=1$.)

In the long run the combinations *HH*, *HT*, *TH*, *TT* will occur in the ratios ap: aq: bp: bq. The respective outcomes to these events are that I receive 2, −2, −1, 1. Therefore on the average at one turn of the game I receive

$$e=2ap-2aq-bp+bq.$$

This we can call my *expectation*.

The game has now been reduced to purely mathematical terms. My task is to choose the ratio a: b (with $a+b=1$) so as to make e as large as possible, and my opponent's task is to choose p: q so as to make e as small as possible.

How do we do this? The whole thing is an exercise in factors!

$$e=(2a-b)p+(-2a+b)q.$$

I have no control over p: q, but if I choose a and b so as to make the brackets equal, that is

$$2a-b=-2a+b=v \text{ (say)},$$

then the right hand side of the equation is $v(p+q)=v$ quite independently of the ratio p: q. The values needed for this are a: $b=1$: 2 and $v=0$.

This means that if I play heads and tails in the ratio 1: 2 I shall break even in the long run, independently of anything which my opponent does. The quantity v is called the *value* of the game and in this case it is zero; the game is fair and the rules do not favour either player. Later on we will examine games in which this is not so, and this method of calculation will then show who is favoured and by how much. Notice that to be sure of breaking even I *must* employ this strategy 1: 2. If I play 1: 1, for example, and my opponent spots that I am doing so then he

can concentrate on tails and my gains will not then compensate for my losses. Likewise if I depart at all from the correct strategy I shall make $(2a-b)$ and $(-2a+b)$ unequal, and my opponent can improve his chances by concentrating on whichever of the two brackets is the least.

Now how is my opponent to play so that he also can be safe? He must argue in a similar fashion, but he will use the rows of the table where I have used the columns. He factorises the other way round.

$$e=a(2p-2q)+b(-p+q).$$

Therefore if he makes

$$2p-2q=-p+q=v \text{ (say)}$$

that is $p:q=1:1$ and $v=0$ then $e=(a+b)v=v=0$.
Thus he can ensure against loss by adopting the strategy 1:1; that is by playing heads and tails in equal proportions.

Now let us take a game whose value is not zero. Consider the penny game played with the table

		B plays H	B plays T
A plays	H	2	−1
	T	−1	1

Calculation on the previous lines shows that $a:b=2:3$ and $v=1/5$. The rules of the game are in A's favour and by adopting the proper strategy he can ensure an average profit of one penny every five goes. (He may be able to do better if B plays foolishly, but we must assume that B understands the theory as well!) Note that this time $p:q$ is also 2:3. To reduce his losses as much as possible B must adopt the same strategy. A and B will always have the same best strategy if the matrix is symmetric in rows and columns, but it is interesting to give a commonsense discussion of this game to try to explain why both players must favour tails in their play. B will give the direct reason that he wins the same in the two cases, but he pays out less when he loses on tails. A has to explain things in a more subtle fashion, he will have to point out that because he (A) wins more on heads B will fight shy of it, therefore his best course is to try to win the small amount more often rather than to take the obvious course of concentrating on the big amount.

Analysis by commonsense soon gets into inextricable tangles of bluff and counter-bluff, and its limitations are soon appreciated if one tries to guess at a solution and then compares one's guess with the calculated answer. As an example try to guess the appropriate strategies for the game with the table

$$\begin{array}{|rr} 2 & -3 \\ -1 & 1 \end{array}$$

The answer is given on page 23.

The same principles are easily applied to the analysis of larger games. A 3×3 game on these lines, which children actually play, is the game of *Stone, Paper and Scissors*. The two participants face one another and, simultaneously, produce from behind their backs a fist (representing a stone), an open hand (representing a piece of paper) or two fingers (representing a pair of scissors). If

the two show the same the result is a draw, but the paper beats the stone (because it can wrap it up), the stone beats the scissors (because it blunts them) and the scissors beat the paper (because they cut it). It is easy to write down the pay-off matrix for the game, but it is more interesting to take a game played with different odds and to decide on the strategies which two players should adopt then. Consider for example the game

		B plays		
		St	*P*	*Sc*
	St	1	−1	−1
A plays	*P*	−1	2	−1
	Sc	−1	−1	3

Again, the table shows the payment which *B* makes to *A*. The analysis is just as before, but with three unknowns instead of two for each player. *A* must adopt a strategy $a: b: c$ $(a+b+c=1)$ such that

$$a-b-c=-a+2b-c=-a-b+3c=v,$$

and these equations have the solution

$$a: b: c=6: 4: 3 \text{ and } v=-1/13.$$

The matrix is symmetric and so *B* must adopt the same strategy.
The game is in *B*'s favour because v is negative.

As an exercise you may like to consider the game

1	−2	0
0	2	−2
−2	0	3

The solution is given on page 23.

These are some of the basic methods of *linear programming*—a branch of mathematics which is being taken very seriously by the United States Armed Forces and by many business concerns who are seeking to apply it to problems of provisioning and the deployment of man-power. Analysis on these lines applies to a wide variety of situations. Williams[3] gives examples on giving medicine to a patient, dating a girl and the dilemma of a man who goes to a football match to sell sun-glasses if it is fine and umbrellas if it is wet (his opponent is the weather!). A number of his examples may be described as variations on the theme of Coppers and Robbers. Here is one.

A has two installations. He is capable of successfully defending either, but not both. *B* is capable of attacking either, but not both. Further one of these installations is three times more valuable than the other. What strategies should they adopt?

We will use "1" to denote the strategy of attacking or defending the less valuable installation, and "2" to denote the strategy of attacking or defending the more valuable. If we regard the values of the two installations as one and three units then the matrix showing the outcome for *A* is

	1	2
1	4	1
2	3	4

This game can be solved in the usual way and it will be found that the solution is a strategy of 1: 3 for *A* and a strategy of 3: 1 for *B*. The value of the game is 3¼. This strategy of 1: 3 for *A* means that three times out of four he should defend the more valuable installation. But how is this to be interpreted? In a war there would be the question of *one* attack only, what is the point of speaking of what to do "three times out of four"? This is a perfectly genuine question and has given rise to controversy. One way out of the difficulty is to say that the military commander must be supplied with a table of random numbers (or some equivalent random sampling device) and decide his strategy by consulting it suitably. In this case he could turn up a number at random and if it ended in four defend the less valuable installation, otherwise defend the more valuable one. You must decide on the value of this answer for yourself.

An obvious question is, "What happens if some of the ratios are negative? Because a negative ratio can have no meaning in this context." Such ratios are indeed without meaning and in these cases we have to employ other methods, which in a strictly logical presentation of the subject would, perhaps, have been discussed first.

A game may possess a *saddle-point*, if it does, then this saddle-point indicates the appropriate strategies. The situation of *Sealed Bids* provides an example. It is not perhaps a *game* in the everyday sense of the word, but it is a game in the sense that the techniques we are discussing will handle it. In this situation *A* and *B* have been given an article whose value is 5 units (say). They agree to settle its ownership by each writing down a secret bid. The highest bidder gets the article on paying the amount which he has bid to the other. (In the event of them both bidding the same ownership is decided by the toss of a coin, the winner of the toss pays his bid to the other and has the article.) The pay-off matrix in this situation is as follows: —

		B bids					*row min.*
		1	2	3	4	5	
A bids	1	2½	2	3	4	5	2
	2	3	2½	3	4	5	2½
	3	2	2	2½	4	5	2
	4	1	1	1	2½	5	1
	5	0	0	0	0	2½	0
col. max.		3	2½	3	4	5	

Remember that the table shows the outcome for *A*. A few specimen calculations will show how it is constructed. If *A* bids "2" and *B* bids "4" then *B* gets the article and pays *A* 4; so the outcome for *A* is a net gain of 4. If *A* bids "3" and *B* bids "1" then *A* gets the article on paying *B* 3. The article is worth 5 so the net gain to *A* is 2. If *A* and *B* both bid "4" then a coin is tossed to decide whose bid succeeds. If *A* wins the toss he pays *B* 4 and gets the article (a net gain of 1) and if he loses *B* pays him 4 and has the article (a gain to *A* of 4). There is an equal probability of either of these events happening so we put the mean value of 2½ on the table.

The analysis now continues as follows. A observes that if he bids "1," "2," "3," "4" or "5" the *worst* that can happen to him is to get 2, $2\frac{1}{2}$, 2, 1 or 0 respectively. These numbers are the least numbers in each row, or the *row minima*, and they are written at the side of the table. The greatest of these minima is $2\frac{1}{2}$. This is given the convenient, if inelegant, name of the *maxmin*. If A bids "2" the least he can expect is $2\frac{1}{2}$. B analyses the columns in a similar way, but since the table gives A's gains and B's losses B looks at the *maxima* in the columns which are 3, $2\frac{1}{2}$, 3, 4 and 5 respectively. These quantities are the most which A can get for the different choices facing B. B naturally wishes to minimise A's gains, and the least of these maxima is, again, $2\frac{1}{2}$. This $2\frac{1}{2}$ is called the *minmax*. Now in this example the minmax and the maxmin are equal, and the entry $2\frac{1}{2}$ in row 2 and column 2 of the table is called a *saddle-point*. If the figures in the table are interpreted as heights in a three-dimensional diagram then the point is a saddle-point in the normal sense of the words. In the games which we have analysed previously there are no saddle-points, the minmax and the maxmin are unequal; their equality is a necessary sufficient condition for the existence of a saddle-point. When a saddle-point exists the best strategy for each player is to play on it. The argument is comparatively simple. By playing on the row through the saddle-point (that is by bidding "2") A has a strategy which ensures that his expectation is at least $2\frac{1}{2}$. Now he cannot do any better than this because the total benefit to be shared is 5 and the situation is the same for the two of them. Therefore he cannot expect to find any strategy to offer better hopes than this, and he has found one (in this case the only one) which will do. B can employ a similar argument. Confronted with this situation the best thing to do is to bid "2".

This game is the same for the two players, but saddle-points can just as well arise in games which are asymmetric. A game may have more than one saddle-point; if it has then it can be shown that all the saddle-points are of the same height and it does not matter on which one either player bases his strategy. The sealed bid situation when the value of the article is an even number will illustrate this.

Another complication which we have not so far considered is the *dominance* of one row or column by another. A simple spelling game provides an example. I choose, secretly, one of the letters *a*, *i* or *o*, and my opponent chooses, secretly, one of the letters *t*, *m* or *f*. If the letters, mine first and his second, form a word I win and he pays me a penny and a further bonus of twopence if the word is a preposition. If the letters do not form a word I lose, and I pay out twopence. The pay-off matrix is

	t	*m*	*f*
a	3	1	−2
i	1	−2	1
o	−2	−2	3

Now look at the game from B's point of view. "*t*" is a very bad choice, because whenever he picks *t* whatever I pick I win as much or more than I would have done if he had picked *m*. It is always better (or as good) for him to pick *m* than *t*. We say that the first column is *dominated* by the second, and if B is never going

to pick t we might as well remove it from the table altogether. This reduces it to

	m	f
a	1	−2
i	−2	1
o	−2	3

But now I notice a similar thing. "i" is never a better choice than "o". (It could be if B was ever going to choose t, but we have seen that he will never do so if he is at all sensible.) The entries in the second row nowhere exceed those in the third, that is to say the second row is dominated by the third and we remove it, reducing the game still further to

	m	f
a	1	−2
o	−2	3

The original methods of analysis now apply to the reduced game and give us strategies 5:3 for each player and a value $-\frac{1}{8}$. The strategies for the players in the original form of the game are thus 5:0:3 and 0:5:3. The value is, of course, still $\frac{1}{8}$ for the second player.

Some games introduce *inoperative strategies*. When some strategies are inoperative the analysis can be difficult and here we cannot attempt to indicate any general theory, but we will merely analyse a game which has actually been played for hundreds of years; so this is in no sense an example which has merely been made up to illustrate the theory. The Italian game of *Morra* is a game rather like Stone, Paper and Scissors in that the players face one another and produce a hand from behind their backs showing a certain number of fingers. At the same time each player guesses how many fingers the other will show. If both guess correctly or both guess wrongly it is a draw, but if one player only is correct in his guess he wins, and he receives from the other a sum equal to the total number of fingers showing. For simplicity we will restrict attention to the version of the game in which one is allowed to show one or two fingers only. Each player then has four possible moves as he can show one or two while guessing "one" or "two." As an example suppose that I show two and guess "one" whilst my opponent shows one and guesses "one". My guess is right and his is wrong so I win further, there are three fingers showing so he has to pay me three. If we denote showing two and guessing "one" by the notation (2, 1) then the four possible moves at any time are (1, 1), (1, 2), (2, 1) and (2, 2), and they give the pay-off matrix

	(1, 1)	(1, 2)	(2, 1)	(2, 2)
(1, 1)	0	2	−3	0
(1, 2)	−2	0	0	3
(2, 1)	3	0	0	−4
(2, 2)	0	−3	4	0

In this game there are no saddle-points and no dominance. What is the best strategy? We will show that there is no single best strategy but we will find one method which will ensure all that a player can reasonably hope. The game is the same for the two players so all that one can reasonably hope is to break even, and it is not as easy as it might seem to find a way of doing this. Let us say at

once that the obvious strategy of randomising the four possible moves in equal proportions will result in a steady loss against a player who understands the theory rather better. We will see in a minute that it is a mistake to play (1, 1) or (2, 2) ever! The best plan is to mix (1, 2) and (2, 1) in suitable proportions. Let us find out what these proportions are. I will adopt the strategy $0: b: c: 0$ $(b+c=1)$ and we will denote my opponents strategy by $p: q: r: s$ $(p+q+r+s=1)$. As before my expectation, the average amount of money I receive each turn, is

$$b(-2p+3s)+c(3p-4s)=(3-5b)p+(7b-4)s$$

where we have put $c=1-b$. If p and s are not zero, remembering that they are essentially positive quantities, we see that this expression is positive provided that $3>5b$ and $7b>4$. That is to say if b is between 4/7 and 3/5 and my opponent is playing (1, 1) or (2, 2) at all I shall gain on the average. This shows that it is indeed unsafe for him to play (1, 1) or (2, 2) at all, and of course the same applies to me because the game is the same for both of us. Further we have shown that by adopting a strategy $0: b: (1-b): 0$ with b between 4/7 and 3/5 I can ensure an average gain if my opponent departs from his best policy. In fact if he adopts the "obvious" strategy $\frac{1}{4}: \frac{1}{4}: \frac{1}{4}: \frac{1}{4}$ I can derive a steady income of about five per cent of the turnover! Notice that if we both adopt the best strategy nobody ever pays anybody anything, and this makes the game rather dull.

Although the Theory of Games has only just been started already it is sufficiently advanced for research organisations to devote to it large amounts of time and money. As yet only the very simplest games will yield their secrets; and to decide the best course of action at any moment in a game of whist or draughts would require a matrix of such enormous size that calculation with it would be out of the question.

The subject is growing rapidly and it is astounding that its invention should have been delayed so long when its foundations are so extremely simple. A new branch of mathematics has just been invented. We can teach it right away.

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[3] Williams, J. D. The Compleat Strategyst. McGraw-Hill, 1954.

[4] Braithwaite, R. B. The Theory of Games as a Tool for the Moral Philosopher: an inaugural lecture delivered in Cambridge on December 2nd, 1954. C.U.P. 1955

SOLUTIONS TO THE EXAMPLES

P.18 : *A*'s strategy, 2: 5; *B*'s strategy, 4: 3; $v=-1/7$.
P.19 : *A*'s strategy, 14: 13: 8; *B*'s strategy, 16: 9: 10; $v=-2/35$.

The Building Societies' Association, of 14 Park Street, London, W.1, publish, free of charge, a publicity pamphlet on home purchasing, entitled *Guide to Saving and Home Buying*, which gives background information on house purchase at about Secondary Modern level. Its arguments can be examined for their logic and plausibility. It contains little or no mathematics but can easily be made to start the questions rolling in about the mathematics of extended credit.

C. H.

MATHEMATICS IN THE SECONDARY SCHOOL CURRICULUM

V.—MATHEMATICS IS THE BASIS OF THE SCIENTIFIC MODE OF THOUGHT

R. H. COLLINS

When finally I come to consider this particular theme I find myself of the opinion, that properly understood, it will provide the most satisfactory aim for the teaching of the subject to all children and in every type of school. I would add that there may be many teachers who, whilst subscribing to this aim, may invalidate it through lack of understanding of the correct function which the subject has in the scientific field. Particularly will this be so if they are at all inclined to have some regard for any of the previous statements on the aims of the subject.

Even with the realisation that the scientific outlook is highly important in the survival of any modern culture it is well-known that it is some time before changes can be seen in their proper perspective in the pedagogical world. Again, it is one thing to comprehend an educational aim, but it is another to find the means whereby it can become a reality in the day-to-day teaching of every school in the state system. In our case the examination bogy is still much with us, despite all pleas to give it up, that this is another factor which may hamper the rapid progress of any changes in the aims of the teacher.

I doubt if it is possible to deal historically with this particular aim because its full significance is so modern in its conception. Certainly it is one which could not have been foreseen in the days of Greek culture, or for that matter, until the beginning of the 20th Century.

If you wish to stretch a point it might be argued that this particular aspiration in the teaching of mathematics did, in fact, originate in the Greek culture. After all, as has already been quoted, Plato said that the value of the subject was that it could be a channel whereby the citizen could be taught to appreciate what was meant by "the good."

Coupling this with the Pythagorean theory that the whole world was "mathematical" in structure and we have the basis of an argument that even in those far off days the Philosophers had an inkling that mathematics would provide a tool for a study of the universe. Of course, it is doubtful if this argument can be sustained for long because it will be remembered that Greek philosophy was always concerned with moral issues and perhaps their attempts to make mathematics work to such ends may have prevented progress in the field of science. At any rate, the idea did not receive much consideration until the 17th Century.

Bacon is one of those who must take some of the credit for the development of this aim for in *Novum Organum* we find him writing this: "Now for grounds of experience—since to experience we must come—we have as yet had either none or very weak ones: no search has been made to collect a store of particular observations sufficient either in number, or in kind, or in certainty, to inform the understanding, or in any way adequate. On the contrary, men of learning, but easy withal and idle, have taken for the construction or for the confirmation of their philosophy certain rumours and vague fames or airs of experience, and allowed to these the weight of lawful evidence. And just as if some kingdom or state were to direct its

counsels and affairs, not by letters and reports from ambassadors and trustworthy messengers, but by the gossip of the streets; such exactly is the system of management introduced into philosophy with relation to experience. Nothing duly investigate, nothing verified, nothing counted, weighed, or measured, is to be found in natural history: and what is observation is loose and vague, is in information deceptive and treacherous. And if anyone thinks that this is a strange thing to say, and something like an unjust complaint, seeing that Aristotle, himself so great a man, and supported by the wealth of so great a king, has composed so accurate a history of animals; and that others with greater diligence, though less pretence, have made many additions; while others, again, have compiled copious histories and descriptions of metals, plants, and fossils; it seems that he does not rightly apprehend what it is that we are now about. For a natural history which is composed for its own sake is not like one that is collected to supply the understanding with information for the building up of philosophy. They differ in many ways, but especially in this; that the former contains the variety of natural species only, and not experiment of the mechanical arts. For even as in the business of life a man's disposition and the secret workings of his mind and affections are better discovered when he is in trouble than at other times; so likewise the secrets of Nature reveal themselves more readily under vexations of art than when they go their own way. Good hopes may therefore be conceived of natural philosophy, when natural history, which is the basis and foundation of it, has been drawn up on a better plan; but not till then."

In assessing the effect of the work and writings of Bacon on the subsequent gradual emergence of scientific thought as a force in the philosophical world, the part played by mathematics must be afforded adequate recognition.

Some indication of this has been well illustrated by Butterfield in his *Origins of Modern Science*. He points out that many important developments which might have materialised in the Middle Ages did not do so because certain mathematical ideas were as then unknown. Kepler's important discoveries concerning planetary motion was only made possible because he had inherited, then developed further himself, the study of conic sections, a study in which he was outstanding in his generation. Only then could the astronomical observations of Tycho Braché become turned into a revolutionary discovery. Later the analytical geometry of Descartes and the calculus of Newton and Leibnitz enabled the problem of gravitation to be synthesized.

Indeed Descartes himself put forward the view that sciences involving forms, shapes and sounds, etc., are related to mathematics "There ought therefore to be a general science—mathematics—which should explain all that can be known about order and measure considered independently of any application to a particular subject."

Such a science he claimed would surpass in utility all the other sciences which in reality depended on it.

Kepler himself said that the mind of man is meant to consider quantity and "it wanders in darkness when it leaves the realm of quantitative thought."

Galileo claimed that the book of the Universe was written in mathematical language and its alphabet consisted of triangles, circles and geometrical figures.

The full impact of these opinions can only be fully appreciated in perspective

and their effects on the educational philosophy were therefore even more delayed. Another factor which mitigated against more adequate consideration of them in the educational field can be associated with the fact that not until the dawn of the 20th Century did the universal education system emerge from anything more than a mere theory.

It is therefore no small wonder that it is not until the 1850's do we find any serious attempt to evaluate this new claim of mathematics for inclusion in the curriculum of every type of school.

I believe that John S. Mill in his *Examination of Sir William Hamilton's Philosophy* sums up the matter with the greatest forethought and clarity: "A mode by which human intellect has proceeded, whereby universal acknowledgment grounded on subsequent direct verification, it has succeeded in ascertaining the greatest number of important recondite truths." For mathematics, through its application to the investigation of the laws of physical nature is a mode "in which the properties of number, extension, and figure are made instrumental to the ascertainment of truths other than arithmetical or geometrical."

Hamilton claimed that the great failing of mathematics was that since it was only concerned with demonstrative evidence it did not teach in theory or practice to estimate probabilities. On the other hand, Mill argued there were two distinct types of sciences (a) those such as mathematics, metaphysics, etc., (b) physiology, geology, etc. The first, which he termed fundamental sciences treated of the combination actually realised in nature.

The abstract sciences (including mathematics) gave no practice in the estimation of conflicting probabilities "which is the kind of sagacity most required in the conduct of practical affairs," but a mathematician who was keenly alive to his defect could still uphold mathematical education, not only as a useful, but also as an indispensable first stage of all scientific education worthy of the name. Mill gave one example of the application of mathematics to the indirect investigation of truth from within mathematics itself, namely, the use of algebra in geometry by Descartes. It provided a way by which number could ascertain the truth about things which were not number. "For the process itself—the deductive investigation of nature, the application of elementary laws generalised from the more simpler cases, to disentangle the phenomena of complex cases—explaining as much of them as can be so explained and putting in evidence of the nature and limits of the irreducible residium so as to suggest fresh observations preparatory to recommending the same process with additional data, this is common to all science, moral and metaphysical included; and the greater the difficulty the more needful is it that the enquirer should come prepared with an exact understanding of the requisites of this mode of investigation and a mental type of realisations.

Up to this time, I may venture to say that no one ever knew what deduction is, as a means of investigating the laws of nature, who had not learnt it from mathematics."

A careful study of these writings and arguments makes it quite apparent that the real value of a mathematical education is that it provides the best mental preparation for everyone who has to live in a scientific age. For none of us to-day can usefully perform his daily task and take an intelligent interest in the world of human affairs unless he is able to know a little about the foundations of science

itself. To do this we must first acquire a scientific frame of mind to know the way in which science looks at life. This, in Mill's view, is impossible save that the person has had prior acquaintance with mathematics.

Now it is possible to interpret this as meaning that mathematical techniques find their application in physics, chemistry and all other sciences, and hence it is only necessary for the study of mathematics to be included in the school curriculum in which all the sciences are studied. If this be so, then provided an adequate case is made out for the educational value of these scientific subjects then the place of mathematics is also assured. In that event the method of teaching the required amount of the subject might be immaterial so long as the necessary information was acquired. Moreover, as stated in the opening quotation of this article, this attitude to the utility of mathematics is not enough ground for all the amount which is now accepted as being a 'must' in any sound syllabus.

A. N. Whitehead in the chapter dealing with the Mathematical Curriculum in his book *Aims of Education* (1932) said that one of the main objects of the inclusion of mathematics in a liberal education is to train pupils to handle abstract ideas and the science constituted the first large group of abstract ideas which naturally occur to the mind in any precise form. The quarrel I have with this statement is the use of the word 'abstract'. By abstract I take it we mean removing anything away from its natural context, and then surely all thinking, whatever the topic, involves abstract ideas. Indeed, sight too might be said to involve the first stage of abstraction since even this is a selective process dependent on the will and interest of the observer. We see those things we desire most and the rest of the world around is ignored. To me 'abstract' can only be used in a relative sense and the subject of mathematics contains many different degrees of abstraction. So too, in their own way, do all the other subjects in the curriculum, and they too may provide an opportunity for the use of the faculty of abstraction.

Perhaps the use of the word is a survival from the past era which considered mathematics to be concerned with a logical thought process which could subsequently be transferred to the field of moral philosophy, an idea which has already been considered and rejected. I have not had the opportunity to search through more recent comments on the interpretation of this utility aspect but I feel that the following quotation from the Spens Report suggests that even in this document the lines suggested above were still being followed. "We have said that we believe that Mathematics should be taught as Art and Music and Physical Science are taught because it is one of the main lines which the creative spirit of man has followed in its development." It then goes on to suggest that mathematical truths have two sides

(a) their applications to the outside world
(b) they can be deduced from previous truths, *i.e.*, the second value is its logical structure.

Putting this in the words of Sir Percy Nunn in *The Teaching of Algebra* (1914), who is quoted in the report, "Our purpose in teaching mathematics in school should be to enable the pupil to realise at least in an elementary way this two-fold significance of mathematical progress. A person to be really 'educated' should be taught the importance of mathematics as an instrument of material conquests and of social organisations and should be able to appreciate the value and significance of an ordered system of mathematical ideas."

In my view the first objective on which emphasis is laid makes too sweeping

a claim. To suggest that mathematics has made 'material conquests' calls for investigation and I am bound to admit that material conquests of any consequence are the realm of the Sciences of Physics, Chemistry, Biology, etc., and the part played by mathematics is indirect and not direct. The second objective suggested is equally vulnerable unless there is true merit in a study of an ordered system for all children, and if there is, there is need for a closer examination before mathematics can have any greater claim over the other sciences to perform this training.

If the case as stated in the Spens Report is a valid one then, too, the immediate result is that much less time in the curriculum need be taken up by the subject, and this is a point which is conceded in the Report. It would further suggest that the subject's educational value has greatly diminished in recent years and there appears to be little appreciation of the true interrelation of the subject to the Sciences.

Having considered this interpretation of the value of mathematics as a mode of thought and comparing it with Mill's statements, I do not think that the two views can be reconciled completely.

It seems to me that not enough attention has yet been paid to the words of Mill or to the part mathematics has played in the development of the scientific world, nor has it been fully appreciated what mathematics is about.

In a few words mathematics is a study in relationships, spatial, numerical and operational, and in this scientific age it is a prerequisite study to any understanding of the Scientific world and present-day Scientific approach to material things.

Any other concrete science is also concerned with a study of the same kind of basic interrelationships as are found in mathematics. Study of concrete science is, however, clouded by many confusing and conflicting probabilities which have, after all careful consideration of all the evidence, to be put into true perspective and an attempt made to establish the true relationship.

In the world of nature man has no control over all the conditions which are likely to influence the experiment of study, whereas in the world of mathematics the conditions which govern a situation can be laid down precisely. As each and every one of these conditions is changed at will, it is possible to examine the unique effect which each one has on the initial relationship. On the other hand, in the concrete sciences of nature the exact and final relationship cannot be known and this renders their study the more difficult. Again, these natural relations are often inexact and of great complexity, yet are ones of general utility for an understanding of this scientific age, so that their full comprehension is not possible to the untrained mind.

No intellect is therefore able to cope with such studies unless it has received prior training in the matter and this is the function of mathematics.

The fact that mathematics is an exact science enables all relationships investigated, and when subsequently discovered, to be fully tested so that a complete assurance of their validity can be obtained without special preparation or experiment. Moreover, by the choice of axioms and the narrowing down of the field to be studied at any one time, the student will have the opportunity to make his first step before proceeding to the study of a higher and more difficult relationship. Again, by a careful grading of the choice of the conditions imposed on the problem, it is possible to complicate the relationship one step at a time so that the pupil will never at any time be out of his intellectual depth until he is adequately prepared to proceed further.

Thus, in the subject of Algebra we might begin by investigating the relationship symbolised in the equation $3x = 18$ and the subsequent method of solving it for x. From here the pupil could proceed to the equations

$$x + 5 = 22$$
$$x - 5 = 22$$

then to

$$3x + 8 = 32$$
$$3x - 8 = 31$$

which combines the first type of relationship with the second. Then could follow

$$4x + 3 = 3x + 11$$
$$4x + 2 = x + 11$$

and after

$$4x + 2 + x = 11 + 2x + 5$$

which again brings into the problem a completely fresh difficulty.

If the maximum benefit is to be gained, however, the teacher must so arrange his approach that the pupil can instinctively perceive the basis of the relationship, without it being necessary for the teacher to explain it. For the moment that a rule is expounded by the teacher instead of by the students, the subject matter has lost its value as an introduction to the study of the world around us.

Thus, instead of suggesting that the equation $3x = 18$ is solved by dividing both sides of the equation by 3 the problem should be approached by the "think of a number" method, "Three times a number is 18, what is the number?" If the child has reached the appropriate numerical level prior to this problem then the child will give the correct answer. Only then is it necessary for the teacher to formalise the work and so clear the way for the next step.

In the carrying out of the objectives suggested, the course should be made to cover as wide a variety of mathematical topics, not only arithmetic, but also algebra, geometry, co-ordinate geometry, calculus, statistics, etc. The exact number of these topics which can be introduced and the depth to which they will be pursued will, however, be dependent on the intellectual ability of the child concerned, but this is a matter of little consequence when compared to the pedagogical approach used in the classroom.

If possible it should be the aim of the syllabus to develop the topics to such an extent that application of one branch of mathematics to another should also be dealt with, *viz.*, the use of algebra in geometry, etc.

It seems somewhat strange that in advocating this claim for the teaching of the subject it may well be said that the very words of Bacon "Mathematics make men subtle" are still true, but of course it is not in respect of the mathematical knowledge with which it treats, but in the way that it teaches man to appreciate the subtleties of the world around him.

I think I cannot do better than conclude by another quotation, again from Mill: " To have an inadequate conception of one of the two instruments by which we acquire our knowledge of nature will lead to an insufficient conception of human knowledge itself as an organic system. He can have no conception of a science as a system of truths flowing out and confirming and collaborating one another, in which one truth sums up a multitude of others."

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BLACKPOOL CONFERENCE, 1958

SECONDARY SCHOOL MATHEMATICS—CONTENT AND METHOD

Sunshine and sea-breezes; good food and a friendly atmosphere. These provided the background to the second annual Conference of the Association held at Redman's Park House Hotel on the week-end of March 14th to 16th. Over a hundred teachers attended the meeting which was arranged this year on a basis of lectures and discussion groups.

Welcoming the guests and introducing the speakers, Mr. C. Birtwistle said it had been gratifying to find so many teachers wishing to come on the course; the demand for places had exceeded the accommodation available.

Mathematics and its Teaching in a Modern Age

Mr. R. H. Collins commenced his talk on this topic by regretting that we were discussing secondary school mathematics in isolation from mathematics at other levels. This he felt was not possible, nor was it possible to isolate the subject in time: the teacher of mathematics in the Modern Age had to look to the future. It was necessary for one to remember constantly one's reasons for teaching the subject.

Mr. Collins believed that our subject was one in which the child could not remain passive in the classroom. Any piece of material used in mathematics should produce a response, and the lack of retention of mathematical knowledge was evidence of non-participation by pupils. If it is complained that a child cannot do a particular type of problem or does not know his tables, are we to go back and attempt the task again, or are we to ask ourselves if we have trained him to think of the subject—mathematics? Here is the contrast of rote learning with a consideration of the child himself. Does a child understand a geometrical theorem when we have proved it to *our* (or the examiner's) satisfaction, but not to his? Mr. Collins suggested that teachers should devote more attention to the vocabulary they used and be certain that the child understood it in all its aspects.

There was, he felt, a need for a change in syllabus; most existing syllabuses consisted of a collection of topics in isolation, whereas the ideal syllabus should enable us to pursue topics to their conclusion. He suggested that Euclidean geometry be scrapped as he felt that this was the only way to improve geometry teaching. Co-ordinate geometry was easier for the child to develop than was Euclidean geometry. Mr. Collins said that the underlying feature of the suggestions he was making was that mathematics should be taught in such a way that the child understood the subject matter and did not approach the subject as a series of rules. This latter attitude was a direct outcome of rigid adherence to a syllabus. He was shocked by the demand of the Secondary Modern Schools that they should take the G.C.E.; their attitude seemed to be one of wishing to tie themselves to a syllabus.

Secondary School Mathematics — The Challenge

Mr. R. H. Fielding commenced his talk by questioning what was meant by its title: the challenge, he felt, was a challenge to all. His intention was to say a few words on his personal attempt to meet the challenge. When he took over his present post he was shocked at the mathematical ability which he found in children of fifteen. Successive teachers had tried to teach the *same* things in successive years of the child's school life and the result was a complete lack of ability in mathematics and a lack of interest in the subject. He decided to take a long-term view, and called together the primary school teachers in his area. At these meetings there had been an exchange of ideas, and each side had benefited. He was convinced that all teachers of mathematics should acquire a knowledge of the whole range of mathematics teaching.

The mathematics staff of a school needed careful selection; the prime factor was that they should be enthusiastic. They should meet together regularly as a group, and the specialists should help the others. It was essential to attempt to make the pupils love the subject for itself. With this end in view he had formed a Mathematics Club and had developed the idea of the Mathematical Family: upon entering the school each child in the first year was "married" to a child in each of the succeeding years up to the Sixth and members of the same family called upon each other for assistance.

Mr. Fielding said that a teacher of mathematics must write his own syllabus which must be more extensive than a mere examination syllabus of an examination board. The syllabus must be constantly modified, and the response would be an understanding and love of the subject by the pupil. But, he concluded, the starting point of successful mathematics teaching is the teacher himself; he must be master

of his subject and he must be willing to learn from his mathematical colleagues at *all* levels. We cannot exist as teachers in a vacuum—we must be interested in children of *all* ages and their development.

In the discussion which followed, Mr. Brown (Staffordshire) spoke of the dislike which some children had of mathematics; this was due, he felt, to the teachers of the subject, but Miss Hersee (Manchester) felt that the dislike was traditional. Mr. C. Hope replied that a recent survey had shown that seventy per cent of school children preferred mathematics to other subjects. Mr. Adams (Assistant Education Officer, Lancashire) said he felt the time had come for L.E.A.s to appoint organizers of mathematics, and this suggestion received considerable support both in this discussion and in the later discussions of the Conference. There was some consideration of the relative importance to be placed on the setting down of work, but Mr. Collins maintained that understanding was of prime importance; once the child understood, the rest would follow. Mr. Beech (Northwich) also felt that one of the difficulties of mathematics learning was comprehension; once this had been achieved we could ask the children to express what they understood and we had to help them with their mode of expression. Mr. Watts (Blackpool) did not agree that Secondary Modern Schools were necessarily wrong in wishing to take an external examination; he felt he had an obligation to his boys to enter them for U.L.C.I. examinations so that they could enter apprenticeship. From the Chair, Mr. Birtwistle said that he was a little surprised that no one had defended Euclidean geometry against Mr. Collins' attack, but apart from one member saying that she felt much could be obtained from Euclidean geometry, there was no further comment from the meeting.

GROUP DISCUSSIONS: "What shall we teach?"

Saturday commenced with discussion in groups on a variety of topics. The group which discussed *Mathematics as one subject—a unified course* were unanimous that it was desirable to teach mathematics as a unified subject. They felt that this could be achieved in three ways. The simple method was by the linking of allied topics (e.g. fractions in arithmetic with fractions in algebra) and this appeared to be the method usually adopted by the members of the group. There was, however, the possibility of unifying the subject by starting from a given situation (e.g. the area of a floor) and developing this in its many aspects (e.g. cost of carpet, scale drawing, trigonometry). A third method that was suggested was by presenting a particular concept (e.g. ratio and proportion) and considering its many mathematical treatments.

The group felt that there was a need for a suitable text-book and suggested that the Association should take some action, firstly, perhaps, by publishing a pamphlet on connecting the topics. On the matter of examinations, all members of the group considered that the G.C.E. syllabus "B" was a great improvement on the syllabus "A", but they felt that new thought was required on the examination papers themselves; the types of question showed little attempt to get away from the typical syllabus "A" question.

In the discussion on *The Secondary Modern Curriculum* the starting point was a syllabus based on utilitarian values: applications illustrating the ramifications of calculation in everyday life. It was attacked on all sides. One had to deal with two

elements in the classroom: on the one hand the child and on the other mathematics. Mathematics could be influenced for better or worse by (i) the external examination, (ii) broad mathematical principles, (iii) the text-book. It was emphasized that broad mathematical principle might be a more wholesome influence than either of the other two. Two members, quick to respond, brought general agreement when they described their experiences. The first suggested one should make up one's own text-book in relation to one's circumstances, resources and the abilities, interests and resources of our children. A school's reputation for integrity was better by far than the stamp of an examining body. The second, a member of last year's Conference, averred that her decision to make the mathematical principles paramount had resulted in happier relationships with the class, keener work and greater progress. Arithmetic should be used as it is needed to further the mathematics which a practical approach would demand.

The Place of Practical Work in the Mathematics Course

The Chairman of the group, Mr. R. H Fielding, posed three questions: (i) Is practical work necessary, and if so, why? (ii) What experiences had the group of practical work in their teaching method which would be of use to the group generally? (iii) What suggestions could be made to improve our teaching by the introduction of practical work? It was the unanimous opinion of the group that practical work was invaluable with children who found difficulty with the subject. Opinion was divided as to whether it would assist the brighter child. Many thought that the empirical approach followed by formal reasoning consolidated the learning processes and resulted in retention of knowledge. A small minority had used practical work in their lessons (i.e. in surveying and the practical constructions in geometry), but apart from this there appeared to be little experience of practical work, and it was left to the leader to outline suggestions: the Cuisenaire apparatus and methods of use; various geometrical aids were illustrated; teaching around a topic (e.g. a bicycle wheel) gave ample opportunity for practical work.

The discussion on *A Pupil-Centred Syllabus* considered the possible interpretations of the subject: (a) the child should be allowed to decide what mathematics should be taught; (b) only that mathematics should be taught which the child could see would be of value in later life; (c) teaching method should be aimed at producing the best response in the child through the presentation of worthwhile topics. In the first part of the following discussion it was suggested that in the Secondary Modern School the syllabus was already child-centred to some extent, since it was varied to suit the different abilities of the children. Most teachers realized that they were teachers of children rather than teachers of a subject. The group discussed presentation of the subject and felt that the main emphasis should be on teaching the child to think. In achieving this the child should not be required to put things into writing too soon. Then it was suggested that as yet we may not know how to teach the subject properly; that investigation might reveal that all children think mathematically far more than we believe and that, if only we could find the right way of canalizing and developing the child's knowledge, we might be more successful.

Examinations

The title of the discussion was considered and lines of development suggested by the questions: do we require examinations at all, internal or external? If we do

not, what—if anything—is required as a replacement? If we do, are we satisfied with existing forms of examinations or do we wish to improve them? Those who seemed to favour examinations stressed that to aim at an external examination provided an incentive to both Staff and pupils and that some such goal was desirable. Alternative papers provided by the Union of Lancashire and Cheshire Institutes were described and also a movement towards experiments in non-verbal types of examination in Staffordshire.

The other point of view began with the question of whether examinations did not hamper the teacher's freedom to choose a broader syllabus. There was much fear in children's minds and our task was to find ways of promoting interest in the subject itself, so that mathematics would become "tools" with which children could do things. To attempt self-generating mathematics was to begin to answer the question of what understanding really means, whereas for most modern school children an attempt to prepare for G.C.E. led to only a dull grind at techniques. Reference was made to notational teaching for the 11+ examination and the need to help our primary school colleagues. Internal examinations and tests were regarded as good by many of those present; they felt that these improved speed and accuracy and the award of marks was a stimulus to the child.

The New Approach to Teaching Method

Mr. C. Hope began his lecture by stating that the accepted view of teaching method was that there were two essentials: (a) a good psychology of learning; (b) a knowledge of mathematics. He believed that what are regarded as the latest theories of learning were paralleled by the view mathematicians have held for the past 2,000 years. Psychology tells us that, when confronted with a field we *perceive* a *pattern* (or *gestalt* or *structure*) within the field. An alternative view is that when confronted with a stimulus, the subject responds with a behaviour appropriate to the stimulus. The S – R response, the Pavlov view of learning or conditioned response, was the way circus animals were trained. Mr. Hope said it was necessary to decide what one hoped to achieve before entering the classroom: a structure or pattern within a field of perception or the correct response to a given stimulus. In his view the latter was better left in the circus ring; children in the classroom needed more than this if they were to learn mathematics.

Mathematics usually starts with a field of elements. Within this field we may see a pattern formed by some or all of the elements. The growth of the pattern is often accompanied by, or described by, a functional relationship. Sometimes a *movement* of an element within the field exhibits certain *invariants* within the field. As we extend the field, we generalise the relationships and the existence of invariants. Thus *pattern, function, invariant* and *generality* are all facets of mathematical activity when confronted with a field of elements (numbers, ideas, relationships, etc.)

In support of what he had said, Mr. Hope then quoted from his report of the Lyme Hall Seminar of 1955 (see "Mathematics Teaching" No. 1). "We start with our pupils who know what they know, which we must find out experimentally." Proceeding from this we provide a simple but pregnant situation which the children can comprehend and by their activity in exploring the situation they are able to move forward. The environment in school is full of mathematical ideas, but it

is important to be certain that what *we* comprehend and discuss is what the *children* have comprehended from the situation. Finally, as a result of each child's findings we polarise the observations round uniformities, order, etc. In essence, said Mr. Hope, the modern method was firstly to awaken interest and then to transform the child's interest in the situation to an interest in mathematics. Mathematics is a language which Sir Percy Nunn had described as facing two ways: outwards to describe the real world, and inwards to create a system of ideas. If we start with the outward-facing aspect it is essential to describe what we see and to discern a pattern which is the framework upon which we can build our system of ideas.

Mr. Hope said that the result of recent research indicated that bright children were better able to develop the "structure" of a principle when given help with *How?* (i.e. method), whereas dull children developed the structure of a principle when given help which led to the development of the structure rather than with *how* a particular problem was solved. However, it seemed the practice in schools to do the reverse, helping the bright children with principle and the dull ones with method. The bright child, in fact, is impatient of long explanations; he comprehends the situation quickly. But dull children are dull, not because they have no intelligence (it is only relative intelligence which is low), but because they do not readily recognize pattern.

The use of practical methods could produce an advance of from one to two years over children taught by the old methods; this was largely the result of the lack of confusion when speaking about mathematics in terms of concrete figures. Finally, Mr. Hope stated what was the recurrent theme of the Conference: if one wishes to help the C and D streams (or for that matter, other streams too) one must concentrate on seeing that they know the *method* of structuring the problems involved and that they understood mathematical principles. Mechanical calculations are best left to machines.

GROUP DISCUSSIONS: "How shall we teach?"

In discussing *Teaching Aids or Learning Aids?* an attempt was first made to define the difference between the two. It was seen that any particular aid could be both, according to the teacher's method of conducting the lesson, but that if there was apparatus that each child could use freely, then "learning" was the more appropriate description. What the title implied was that we had the possibility of a new approach to Teaching Method. Not many of the participants had experience of using aids, but during discussion it became clear that two functions were involved: some dynamic material made a bridge to mental appreciation of mathematics, i.e. it helped insight into new concepts; other aids, of practical applications, were concerned with posing problems on what was already basically understood.

Brief mention was made of some uses of Cuisenaire rods and geo-boards as examples of the advantages of multivalent aids which were simple for children to understand and yet could help towards deeper understanding of many topics of algebra and geometry which children may not have mastered from blackboard and text-book: that is by manipulation of symbols and isolated diagrams. Some aids were very simple, e.g. curtain rings for showing overlapping and touching circles. Two or three pieces of stick could be used to give dynamism to all the properties of the geometry of straight lines, angles, triangles, etc.

Opening the discussion on *Relating Mathematics to the other subjects of the Curriculum*, Mr. Collins proposed that consideration should be given to the question of correlation from three viewpoints: (a) selecting a subject around which to build the mathematics; (b) teaching mathematics solely as *mathematics* but referring to other subjects for illustration; (c) leaving other subjects to illustrate mathematics. It was immediately suggested by one member that since many aspects of certain subjects are dependent upon mathematics for interpretation of their meaning, there could not reasonably be any clear line of demarcation between the two. Supporting the view that, in general, more correlation is needed than at present, several members who teach other subjects beside mathematics agreed that mathematical standards of pupils are generally inadequate in subjects outside the mathematics lessons.

The view was expressed that mathematics should be related to every subject; mathematics teachers should teach basic skills in mathematics, at the same time making the pupil realise that he was going to use these skills in other subjects. Hence, practical situations could be found within the school life which could make mathematics "real" and at the same time assist in the teaching of all subjects. But those teachers to whom mathematics appeared as an end in itself believed that the mathematics teacher who was not concerned whether his mathematics was used in other subjects, tended to teach more intensively, and therefore more effectively.

The prime difficulty of the group which discussed *The use of film and film-strip in mathematics teaching* was that few had had experience of the use of these media. In fact none had seen a mathematical film, and it was left to the group leader, Mr. Birtwistle, to describe some typical mathematical films; he made particular reference to those by Nicolet on geometry and films by Fletcher on more advanced topics. The group, however, soon established the advantage of the film over the film-strip in that it made the subject dynamic; it was pointed out, however, that a film-strip could be made dynamic by the simple expedient of moving the projector. The group decided that the advantages of using film and film-strip in mathematics teaching were that they clarified the teaching, they were time-saving (especially where complicated diagrams were concerned), they could be referred to at any time during the lesson, and that they had a stimulating effect on the class by virtue of their vividness compared with, for example, the blackboard. The group felt, however, that these methods had to be used with discretion and the success depended on the children and the teacher as much as on the individual film or film-strip; their use must be complementary to good teaching. They felt, too, that there was a disadvantage in that these media are developed and produced by one person and used by another who may have completely different ideas of what should be included in the lesson. This led to the consideration of the possibility of making one's own material. The production of a film was realized as being costly and difficult, but anyone possessing a 35 mm. still camera (a size becoming increasingly popular among amateur photographers) could easily produce a film-strip for his own use. The group concluded its discussion by considering possible future topics for mathematical film-strips: similarity, fractions, metric system, solid geometry, growth and form, and architecture.

In discussing *Running a Mathematics Club* it was stressed that in running such a club it was essential for the leaders to have a clear idea of the aims behind

its formation, and a fund of ideas for its future activities. It should be properly constituted with a child Chairman and Committee which would meet regularly to arrange programmes. The first aim was to stimulate an interest in mathematics and to do this a number of useful projects could be attempted and exhibitions arranged for non-members of the Club to see. Short talks might be given by the children on research they had done; the history of mathematics makes a good starting point here. For members who were good at wood and metal work, a useful activity was the manufacture of aids which could be used in lessons.

Graph, Gradient and Function Concept. A short lecture was delivered by the group leader, Mr. Hope, which provided much food for thought. In brief, in many mathematical situations two classes of numbers are related. These are said to be functional relationships and may be classified as linear, quadratic, etc. If we reconstruct the syllabus so as to develop each of these relationships in turn, we shall unify all topics of mathematical analysis bringing out Whitehead's Trinity of Variable, Form and Generality. For example, we teach first topics about linearly related variables. In each case we calculate tables of values, draw graphs and deduce formulae. Examples quoted were: *cost = number × price; circumference = Πd; distance = speed × time; extension = k × load; etc. Cost = number × price* would lead first to the compilation of a ready reckoner, then the drawing of graphs, and then to the formula which each graph represented. By drawing the graph of numbers of articles at *1s.* each, then of articles at *3s.* each, and asking where the graph of articles at *2s.* each would be found, the idea of gradient is begun. With suitable ordering of topics there is a gradual growth of the concept of gradient, of continuity, of formula, etc., until a stage is reached where a summarising series of lessons would lead to all the properties, etc. associated with $y = mx + c$, including simple and simultaneous equations. A similar treatment would be extended to the quadratic, including the rate of growth and transformations of $y = x^2$ to the general case $y = ax^2 + bx + c$.

GROUP DISCUSSIONS: "Some Problems of Mathematics Teaching"

The group discussing *Aspects of Geometry* commenced by considering the problem of how to make a reality of area and volume properties. It was felt that the concept of area must be established before any real progress could be made when dealing with the area theorems and there was much discussion on methods of introducing areas, including tracing irregular shapes on to squared paper and counting the squares, and finding areas using a unit other than a square. Examples were given of work on a dynamic approach in which the whole set of triangles, etc. was given in quick succession and from which the properties of invariance were seen intuitively, and special cases noted with the reasons why they appeared. There was general agreement that children do not fully grasp the meaning of words, e.g. perpendicular lines must be vertical; any line which meets a circle *touches* it provided it is not drawn long enough to cut the circumference. It was felt that as teachers we do not spend enough time discussing what we mean by certain words, and at times even foster wrong ideas by always drawing figures in a certain way. It would seem that children have a natural desire for symmetry and regularity in their diagrams, but no conclusion was reached as to means of persuading them to draw a special figure only when it is demanded by the problem under discussion.

Sixth Form Mathematics. This specialised discussion group began with references to "pre-Sixth Form" work. It was obvious that most schools adopted one method or other to ensure that some Sixth Form work was done in the Fifth year by the brighter pupils. Generally an Ordinary level examination (Syllabus II or Ordinary Alternative) was chosen for these forms so that the beginnings of Co-ordinate Geometry and Calculus could be taught. So far as text-books were concerned it appeared that there was no single book which covered all topics adequately and most teachers were using five or six different books in the course of the two years. A survey of teaching periods revealed the grave difficulties experienced in many schools (especially girls' schools) where frequently a Single Mathematics Group and a Double Mathematics (e.g. Maths. and Theoretical Mechanics) Group were taught at the same time. In the worst cases of shortage of staff it even meant First and Second Year Groups being taught together.

The question of how to prevent undue specialisation on G.C.E. Syllabus work produced many suggestions including: (a) a termly essay on some mathematical topic; (b) a good mathematics section of the School library (*without* text-books); (c) some science and mathematics lessons should be given in a foreign language!

Under the heading *Sociological Studies and Statistics* there was a long and lively discussion on the contributions mathematics was able to make to the social sciences. As examples of the presentation of data in graphical form, histograms, bar graphs, "pie" diagrams, isotype, and tables and charts of all descriptions were quoted, the data for which could be obtained from the Geography department (weather, output of materials, imports, etc), the History department (unemployment, prosperity, etc.), from measuring the altitude of the sun, from road censuses, rural studies, etc. In all these it was necessary to make the children think, to allow them to suggest their own censuses, and try to introduce the note of enquiry. As far as consideration of data was concerned, a time and motion study of the domestic science department was suggested as a possibility, taking the number of times different movements were made and an analysis of the results. Height and weights would provide material for averages, spread and correlation by considering simple scattergrams. The Normal Probability curve should be recognised and its general form studied. This would involve a very simple consideration of simple binomial probabilities. A study of local markets would involve different prices, periodic price fluctuations, hire purchase, etc. It was pointed out that in many topics graphs would illustrate the mathematical principles which govern the economic principles involved in a topic. Finally, if one took Commercial Mathematics in G.C.E., an incentive would be provided.

So great was the demand to discuss *Difficulties in the Secondary Modern School* that two discussion groups had to be formed to consider this topic. At the beginning of the discussion by the first group, some of the many points put forward were the difficulties of large classes, cramped accommodation, poor allocation of time to the subject on the school time-table, lack of aims of the school and the co-ordination of work within the years, the discipline of the children and the apathy of some parents and their attitude to school work. It was felt that under the latest Burnham arrangements a "Head of the Mathematics Department" should be appointed, even in the smallest school, to take charge of all mathematics through-

out the school. He should be responsible for the schemes of work within the subject, co-ordinating the work at every stage, and holding regular meetings of the Mathematics Staff to discuss syllabus content and method. Whether the G.C.E. should be taken in Modern Schools and the syllabuses designed to this end, was not resolved, but it was generally felt that the children should have an incentive.

The second group started with similar statements of problems: the difficulty of having non-specialist staff to teach the subject when they had no interest in it; the frequent changing of teachers; classes split between two or more teachers for their mathematics lessons. Here, too, the external examinations weighed heavily in the minds of some teachers. The problems were clearly too great to be resolved in an hour's discussion, but the group had been asked to consider more fully what the Association and the North-Western Group could do to assist teachers with their problems. Many felt they would like to see demonstration lessons with children of this age-range making use of new materials (e.g. Cuisenaire rods). A film-strip viewing session was also considered as being of likely value and a suggestion was made for a general discussion on aspects of work in Secondary Modern Schools introduced by a panel of teachers in such schools. (This latter suggestion is being taken up for a meeting in Manchester in October.)

Ain't Maths Wonderful?

Heading the bill for the summer show at the nearby North Pier was David Nixon and we seemed to be having a foretaste of this when our last session commenced on Sunday afternoon with jokes and card tricks. But, unlike the conjurer's tricks, these were not "all done by mirrors" (although a selected card mysteriously appeared on one of the wall mirrors in the Conference Room!). These, explained our tame trickster, were all done by mathematics, and went on to show the mathematics involved. But now the Grand Illusion: the trickster became a Solomon as he divided out an inheritance so that every one of the benefactors received more than what he considered his fair share! And thence by magic carpet, first to the land of twelve fingers and then to the land of only one finger where the inhabitants have only two digits, one and nought. But our Aladdin showed them at work dashing around at 186,000 miles per second doing endless calculations in an electronic computer. If this life was getting too complicated we could get back to Nature—via the mathematical route, of course, stopping at the Pascal Triangle and the Curve of Natural Frequency.

And so to the Fibonacci Series. And now our wizard produced a plant (looking suspiciously like a maltreated copy of the *Manchester Guardian!*) and counted the leaves and their revolutions as they spiralled round the stalk. And lo! they were successive numbers of the Fibonacci Series. Then odd and even ratios of the terms from this series were taken and boiled down in the wizard's pot to a limit of ·618034—a magic number indeed, for we found it was the ratio of the Golden Section. Now the chase grew fast and furious: we found the ratio in Classical Art; from a rectangle with sides in this ratio we took a square and found the remaining rectangle with this same ratio, and we repeated the process ad liberandum. Then we described a curve in the figure and found it a logarithmic spiral. And so back to Nature again with a sunflower head with seeds arranged in spirals, and the ratio of the numbers of spirals in the two directions as successive terms of the Fibonacci Series.

Next came other spirals. Then Prospero waved his wand and we dived "full fathom five" to take a look at "the oldest living fossil"—Pearly Nautilus. But Pearl's maiden name must have been Fibonacci because her vital statistics disclosed her shape to be a logarithmic spiral. And now the magician's wand moved faster and faster describing for us all manner of fantastic curves; then Hey! Presto! we were on the time machine. Back we travelled to see the Egyptian Rope Stretchers at work—and not using a 3: 4: 5 triangle either! That misconception, said our guide, was caused by the illogical argument of one called Cantor. We shook our heads in sorrow over such misdeeds, only to be shown that we had indulged in similar illogical argument a brief half-hour ago. And so, suitably chastened, we waited for our magician to lay down his wand and close his book of spells written in their strange mathematical hieroglyphics. For, said he, as he discarded his robes, this was all done by mathematics! Then, doing his last illusion and transforming himself into Mr. Birtwistle who had organized the week-end course for us, he said that—from what we had just seen and, even more, from the week-end as a whole—he hoped we would agree with him when he said "Ain't Maths. Wonderful?"

The Conference closed with a vote of thanks to Mr. Birtwistle for organizing the course, and to all the members of the team for their splendid work as speakers and discussion group leaders. There was a strong expression of opinion that in the future Blackpool should see another of our week-end Conferences.

Report edited by C. BIRTWISTLE

A SHORT COURSE FOR SCIENCE AND MATHEMATICS TEACHERS

A short course for teachers of science and mathematics was held at University College London from April 14th to 17th, 1958. The course was organised by the Shell Petroleum Co. and was attended by about 250 teachers.

At an informal reception on the first evening, Mr. F. J. Stephens, a Managing Director of Shell, explained the purpose of the course which was in some measure to bridge the gap between school and industry. A similar course had been held in April, 1957, but in the present one the emphasis was more on the part played by mathematics in the science and technology applied to industry. It was made clear that the course was not meant to be narrow in content, neither was it intended to enlist the teachers as scouts to lure their Sixth formers into the Shell organisation, but the petroleum industry would form the background to the course.

The main substance of the course consisted of eight lectures on the next two days. Professor D. McKie of University College lectured on the *Development and Organisation of Modern Science*, tracing its development from the national scientific academies of London and Paris, and with particular reference to the scientific papers and periodicals of the time.

During the coffee break the attendants showed their efficiency by completely dismantling the microphone and loudspeakers which had been used in the first lecture, though this was perhaps not entirely appreciated by those at the back of the

large lecture theatre. However, luckily for us Professor Bruckshaw of Imperial College had no need of aural aids in his stimulating lecture on *Mathematics as a Tool in Oil Exploration: Geophysics*. He dealt with the various ways in which the land structures where oil might be present could be ascertained. Mathematics was used in the interpretation of gravity, magnetic and seismic surveys, as well as in ordinary aerial surveys. The techniques used depended on potential theory or wave propagation and the statistical nature of the date must also be taken into account. The many questions at the end of the lecture showed the great interest aroused by Professor Bruckshaw.

The afternoon lectures were both given by visiting lecturers from abroad. The first on *Chemical Engineering in the Petroleum Industry* was given by Professor Ir. H. Kramers of Delft University of Technology, who emphasised that in studying the processes involved in chemical plants, it was most important to study the relatively few important underlying principles.

It was unfortunate that Dr. Biot's lecture followed without a break, as we were flagging a little by then. Dr. Biot, from America, lectured on *The Function of Mathematics in our Civilization*, but he kept rather too closely to the notes which had been distributed and his lecture added little to them.

The next morning saw a lecture on *New Chemicals from Petroleum* by Mr. Iliff of the Shell Company, and this was followed by a lecture by Professor D. C. Christopherson of Imperial College on *Mathematics for Engineering Students*. In a forceful and often amusing lecture Professor Christopherson stressed the necessity of keeping mathematics as a separate subject in the curriculum but of utilising it as a unifying influence. He pointed out the advantage of the early introduction of numerical methods and suggested the use of analogues even at the Sixth form level as an aid to understanding.

Without doubt the highlight of the course for most of those present was the lecture on *Mathematical Techniques in Operational Research* by Dr. Vajda of the Admiralty Research Department. Dr. Vajda's enthusiasm was communicated to the whole room, and if there were some who did not follow all his examples in detail they felt that he had managed to convey the spirit of the new techniques. Dr. Vajda's lecture was also an excellent example of the correct use of visual aids, not for their own sake but to emphasise and drive home a point in a compelling way. He used models of intersecting planes to illustrate one solution and an excellent film-strip he had construc.ed in order to demonstrate queuing theory. Altogether an outstanding lecture.

The final lecture by Mr. Kirby of Shell served as an introduction to Stanlow Refinery and Thornton Research Centre which were visited by most of the party on the next day as conclusion to the course. Throughout the visit the representatives of the Shell Petroleum Co. were most helpful and made the visit an enjoyable one.

Looking back on the course, what were the outstanding features? Dr. Vajda's lecture, and those of Professor Bruckshaw and Professor Christopherson; of the films shown after the lectures *The Rival World*; and finally, in the lavish hospitality dispensed by Shell, a glimpse into the "expense account" world of business, a field unknown to teachers.

I. C. N. BELL

MATHEMATICS THROUGHOUT THE CHILD'S SCHOOL LIFE

Are secondary-school teachers aware of the problems of teaching mathematics to younger children? Some are content to disparage and express the view that it would be as well if no mathematics were taught in the primary schools. They would not then have to spend time in reteaching at the secondary stage, or in overcoming well-formed antagonisms.

What is the primary teacher's view-point? Is it possible that the secondary stage teachers do not know because the primary school teacher has himself only a hazy and limited idea of his objectives and obligations?

What, in fact, are the duties of the primary and secondary schools to the child's mathematical education? The answers, which are too complex for glib solution, need to be discovered and will not be attained until teachers in each category share, with true humility, their experience and their philosophy. The administrative break at eleven-plus is only too obviously the occasion for a gap in understanding which the teachers acquiesce is perpetuating.

The one-day conference at the Dick Sheppard School on February 22nd had an importance over and above the success of the demonstration lessons which formed the core of the meeting. It was the beginning of an attempt to bring together for mutual benefit teachers at primary and secondary level. As a start to the sharing of information it had great potential value.

The demonstration lessons provided a contrast in approach. Mrs. R. M. Fyfe gave a lesson to a class of nine-year-old children in which she used geo-boards to clarify the distinction between square and diamond. With a group of six-year-olds, Miss J. Clarkson sketched rapidly the number of properties and operations of addition and subtraction with Cuisenaire rods. Finally, Miss D. Bass organised group practical work with secondary-school girls, providing each group with an assignment (based on the mathematics of the triangle) suited to the age and ability of its members.

The first lesson aroused a good deal of discussion. Considerable conflict of opinion about the depth of understanding achieved by the pupils was evident. Some deplored the absence of a formal definition of the figures involved; others countered with the view that understanding having been achieved, verbalisation was not essential until the pupil desired it. To some, the lesson was valuable time wasted, and to others, an exposure of a lack of understanding in the child frequently unnoticed by the teacher. The discussion showed clearly that those who thought the lesson largely wasted would have had different aims in the same situation, and failed to judge accurately the end-product because they were not in sympathy with Mrs. Fyfe's aims.

At the end of the conference, Mr. R. H. Collins expressed the Association's thanks to the Headmistress of the Dick Sheppard School and to all who had contributed to the success of the occasion.

D. H. W.

FIFTH ANNUAL GENERAL MEETING

The meeting on February 22nd, 1958, opened by the reading of the President's message.

The customary reports on finance and activities included the statement that membership had doubled during 1957, the roll at the end of the year numbering 849. An appeal was again made for the doubling of membership during the current year. Three copies of the bulletin would be published.

The names of the Officers and committee members who were elected are printed elsewhere. Mr. A. Ivell and Mr. D. H. Wheeler are welcomed as new recruits. All those who served during the past year were thanked for the work they had done for the Association.

Under any other business, the following motions were proposed, seconded and carried *nem con*:

(a) That the Committee be given power to raise the subscription to 10s. for 1959.

(b) That full expenses be paid to committee members for attending meetings.

R. M. F.

CONFERENCES

October 4th, 10.30 a.m. to 4 p.m.

The Middlesex Branch extends an invitation to all members and their friends to visit Mountgrace Comprehensive School, Potters Bar. There will be a demonstration lesson with a film by Mr. R. H. Fielding; reports of original work done by members of the group; discussion of methods of teaching some topics usually regarded as difficult.

Particulars from the Secretary.

October 11th

Mathematics and the Primary School. The meeting is being organised by the Department of Education of the University of Oxford.

Particulars from Mr. A. Ivell, 6 Cheriton Road, Aylestone, Leicester.

November 14th and 15th

Mathematics in the Secondary School, at Cardiff.

Particulars from Miss Y. B. Guiseppi, 34 Pollard's Hill South, London, S.W.16.

November 29th

Mathematics for ages 5 *to* 15, at Reading.

Particulars from Mr. A. R. Colyer, Ashmead School, Northumberland Avenue.

MIDDLESEX BRANCH

Chairman: D. H. WHEELER

We are attempting the formation of an active group who meet in order to stimulate and encourage each other to try out new methods and to reconsider the syllabus and the reactions of children in the process of learning.

In November, 1957, Mr. Ian Harris talked on dynamic proofs and methods in geometry, which incorporate the concept of *limit* from the earliest lessons. Uses of geo-boards were demonstrated in the afternoon. At the meeting in March, algebra was considered as "operations on operations and the dynamism of equivalences", and it was shown that Cuisenaire rods form an invaluable tool. The May meeting included a demonstration lesson which showed without doubt that ten-year-old boys take instantly to the rods and can use them with ease for fractions and for permutations. The syllabus of Mountgrace Comprehensive School was explained in a lecture by Mr. R. H. Fielding: and a student of King's College presented original teaching aids.

The next meeting will be held at Mountgrace School, and visitors will be welcome.

WEST RIDING GROUP

There has been an unavoidable change of date for the next meeting of the West Riding Group, and members are asked to note that it will now be held on Saturday, October 18th, 1958, at Nightingale Primary School, Doncaster.

In conjunction with the West Riding Group the Institute of Education, Sheffield is arranging a short course of about six weekly lectures on *The Child's Concept of Number*. It will begin in October, 1958 and will be held in Doncaster. The series will include not only reference to Piaget's work but also talks by experts on a number of different methods of teaching number.

The course should appeal to teachers in secondary schools as well as those in primary schools, and anyone interested should apply to the Secretary, Institute of Education, Sheffield, or R. Shaw, 23 Pipering Lane, Scanthorpe, Doncaster.

A.T.A.M. FILM UNIT

Accounts for period 13/1/57 *to* 24/1/58.

Income	£	s.	d.	Expenditure	£	s.	d.
Balance in Hand	33	13	9	Lighting Equipment	9	18	4
Donation	2	10	0	Switches	1	7	2
				Miscellaneous Equipment ...	6	7	4
				Balance in Hand	18	10	11
	£36	3	9		£36	3	9

Signed T. J. FLETCHER,
Director

Summary of full accounts, which were audited, found correct and signed by Ian Harris and R. H. Collins on February 2nd, 1958.

BOOK REVIEWS

LET'S BE PRACTICAL. Alan J. Gould. Chambers. 30 cards and Teacher's manual, 4s.

At last, a system of work cards for the Secondary Modern School! 60 tasks are set, two to a card. The teacher's book has a list of apparatus to be prepared and the pupil is able to carry out a task using his hands as well as his head. The tasks are mostly concerned with the arithmetic of everyday life. Bills, postage, time-sheets, railway guides, weights and measures, calendars, measuring and drawing triangles, etc. and some puzzles with numbers.

At first sight it looks like a thin text-book printed on manilla and much of it is, but much of it can be made practical.

Cards have the advantage that their completion gives a feeling of achievement that a book does not. A card on H.P. for example can base its work on catalogues collected by the pupil and there is thus a feeling of working on location. The teacher will find them invaluable in remedial work but will deplore the fact that many opportunities are lost and many gaps remain to be filled.

It is a pity that most of the cards are concerned with economic arithmetic. Could we not have some which would concern themselves with the arithmetic and mathematics of science and astronomy and surveying? Could we not depart from the text-book tradition? Could we not have cards which would lead to the collection of experiences out of which some mathematical structures could develop?

Messrs. Chambers are to be commended on launching such a venture, it is to be hoped that improvements and additions will be forthcoming.

C. H.

NUMERICAL TRIGONOMETRY. R. Walker. Harrap. pp. 184, 8s. 6d.

This book gives a comprehensive treatment of the basic geometrical principles underlying trigonometry, covering a fairly wide field in a very small space. The treatment is direct but tends to feed information rather than to provide opportunities for the pupil to discover and to establish the ideas behind trigonometry.

The brevity of the bookwork has made possible a generous collection of examples of varying kinds. The examples are very well graded with sufficient repetition. It is careful not to go beyond the requirements of "O" Level.

Its graphical treatment is a disappointment to the reviewer; for although the graph of sin x is constructed for values of x less than 90°, yet no comparison of sine and cosine is made and the results of adding sine and cosine functions to produce square waves, pulses, etc. is not treated: surely a grave omission in a book intended for the last half of the twentieth century.

All the usual topics are there but there is one section on introducing the measurement of the earth. What a pity the author did not take the opportunity of amplifying this section to include a little elementary astronomy of the earth? Certain readers of this review will be very disappointed to hear that radians have only one brief mention when areas of sectors are touched upon.

Definitely a book for those who must pass G.C.E. at "O" Level; perhaps of value for those who introduce trigonometry at first form level; but its value would have been enhanced had more been introduced that had a bearing on other branches of knowledge and of human activity.

I. L. CAMPBELL

ALGEBRA, Part II. V. M. Bradis, N. S. Istomina, A. I. Markushevich, K. P. Sikorskii. Uchpedgiz, Moscow. 1957. pp: 340. 3·40 roubles.

This book caters for the eighth, ninth and tenth classes in the secondary school, with an age range of about 14 to 17. Since the less academically gifted children tend to go to vocational schools from the age of 14, it may be assumed that the pupils for whom the book is designed corresponds roughly to the fourth and fifth forms of an English grammar school, together with a lower sixth in which there is no specialisation and into which the great majority of the fifth proceed. The range of topics includes roughly what an English science sixth would have covered by the end of the first year if they did not do Additional Mathematics at O Level; there is rather less of manipulative technique but decidedly more by way of fairly rigorous treatment of fundamentals. We find, for instance, the formal definition of a limit, and definitions of operations on and functions of real numbers considered, when irrational, as the limits of their decimal approximations. There is a good chapter on complex numbers, which are introduced by generalising a vector representation of operations on real numbers. Differentiation is treated on familiar lines, but integration is altogether omitted. There is a thorough discussion of linear and quadratic inequalities; it is therefore surprising to find that the only method given for distinguishing between a maximum and a minimum is to substitute neighbouring values of the argument. In general there is little to criticise on technical mathematical grounds, but one other point seems worth mentioning. Logarithms are introduced only after a discussion of the exponential function of the real variable by the method indicated above. This involves the evaluation of, e.g. $10^{1\cdot4}$ as an approximation to $10^{\sqrt{2}}$. "To obtain its value," we read, "there is no need of a special computation: there exist tables of the values of numbers of the form 10^x." Really, this will not do!

However, it is fair to suppose that faults of this kind are of the nature of growing pains: for it appears from the preface that mathematical teaching in Russia is in the early stages of a movement in the same direction as that which has so greatly changed English teaching over the past half-century. This book is formal by contemporary standards; but its definitions and theorems are invariably introduced and expounded by very full and thorough explanations with adequate examples. Since the book contains no exercises, which are provided in a separate volume, it will be clear that the explanations occupy far more space than is customary with us. In fact the book approximates much more closely to an extremely thorough teach-yourself book than to the conventional English school book. It would be interesting to know just how it is used. Most English grammar-school teachers would probably agree that their pupils are not very good at learning from the book. But may it be that they would be better at it, and would develop earlier the power of independent study, if our books were more like this? If its contents are indeed successfully assimilated by the range of pupils indicated, the question is certainly one that we ought to consider.

There are some interesting historical notes, from which one quotation, not typical, may be given: "The turning point in mathematics was Descartes's idea of a variable. Thanks to this, movement and dialectic entered mathematics."

E. BARTON

NEW REFORM ARITHMETIC. W. J. Kelly. Macmillan & Co. Ltd. For Standard I, pp. 44, 1s. Drill Book for Standard I, pp. 16, 1s. For Standard II, pp. 56, 1s. 3d.

The publishers claim that this series marks a forward step in simplifying and popularising the study of arithmetic. We have received the first two of seven books apparently written for Northern Ireland. To an English teacher, they would appear to be drill books graded very well and arranged on the theory that repetition produces results. Book I contains very little reading matter but deals with whole numbers and shilling and pence. One novel feature is that of putting units at the head of an addition column:

$$\begin{array}{r} \text{dens} \\ 46 \\ 9 \\ 7 \\ \hline \\ \hline \end{array}$$

The numbers are restricted to those below 36 and multiplication tables as far as 6×6 are included.

Book II continues the experience of numbers to $12 \times$ and restricts money to sums less than £1 although answers may be greater than that amount. More reading matter is included.

The books should be valuable supplementary drill material for primary classes having a more enlightened mathematical course or for backward classes in the Secondary Modern School. They are definitely tools for the teacher to use and would not embarrass him, no matter what theory or principle he uses to conduct his classes.

C. H.

FLATLAND, by "A Square" (E. A. Abbott). Basil Blackwell. pp. 102, 9s. 6d.

Flatland must be one of the earliest examples of space fiction, and the ideas on which it is based have been employed in so many popular expositions of relativity that we forget that the book predates the theory by many decades. The publishers earn our thanks by giving us the chance of reading it again and putting an established classic of light mathematics on our shelves at a most reasonable price.

It is a strange, original and amusing book, but reading it nowadays is not as easy as it was since it is social satire every bit as much as mathematical fantasy and many points are easily lost on the modern reader. We know that the author was a Cambridge classic and a distinguished headmaster, but what were his intentions in writing *Flatland*? Was he really the class-conscious misogynist he so easily appears, or was he writing with heavy irony which can be missed after this interval of time? What is the significance of an episode such as the colour revolt? Is it the mathematics teachers who have banished colour from Flatland and is teaching practice being pilloried? If so what reforms did he desire? Not knowing the smaller details of the social history of the time it is difficult to disentangle the two strands of social and mathematical satire which form the book; for the strange world of Flatland is not to be compared with the universal nonsense of the self-contained Wonderland of Alice, it is rather more a Lilliput or an Erewhon

whose significance can only be seen fully in the relation it bears to contemporary England. The Eighties are long passed and few of Abbott's friends can now remain. Can someone, in a brief essay, preserve for us the man and his purpose before it is too late?

T. J. F.

TEACHING MATHEMATICS IN AN EXPANDING ECONOMY—1. C. Gattegno. Cuisenaire, pp. 20, 2s. 6d.

This pamphlet, which has nothing special to say about expanding economies, has a lot to say about the teaching of mathematics and the training of mathematics teachers. Much of what is said is not new nor is it original; the reader will find many generalizations unsupported by evidence; but as usual in a book by Dr. Gattegno there is plenty to think about.

That classes should be given problems to solve (Dr. Gattegno's "functional syllabus") rather "than to struggle to discover the relation of some fact or idea to their own experience" has been propounded for a long time. It needed no fancy name when the reviewer was trained before the war: it was just labelled good teaching. It is a pity that the word "functional" has been given yet another meaning so that it is almost impossible to use it any more without ambiguity. There is a good deal of "looking back in anger" to the past, to deny the relevance of "the authority of venerable writers", but can we avoid consulting authority for the canons which guide us in selecting the right experiences from the infinite variety confronting us? Dr. Gattegno renews his claim that mathematics is not linear, but unless there is some organisation of the thinking into a logical order then there may have been mathematical activity but it does not follow that mathematics has been comprehended. In fact the situations to be discussed are selected by the teacher to conform with an unfolding development of mathematical knowledge, strikingly confirmed by the author when he insists on experience with discrete or finite objects. Is there a transfer to the continuous?

Primary teachers should read what Dr. Gattegno has to say about their task. Many of my aquaintance will find the language and ideas quite new to them but that is because in the Primary School, whilst we tolerate literature at an adult level, mathematics is almost completely ignored in the urge to calculate; Dr. Gattegno suggests a remedy in a "revolution in the training of primary teachers". His revolution is perhaps more in the topics he suggests than in the methods he advocates. Were he to visit training colleges, he would find many who were confronting students with real situations to "mathematize" about. It is a pity that all his examples are abstract, but then I suppose that if we are to teach mathematics we must be abstract!

It is a provocative little pamphlet; good to read to knock down; good to read to stimulate; but nevertheless good to read.

C. H.

Solution to problem on page 9. If the sides are $2x-1$, $2x$, $2x+1$ then $3(x^2-1)$ has to be a perfect square. Numbers with this property satisfy the recurrence relation $u_n=4u_{n-1}-u_{n-2}$, and the values of x are the terms of the sequence 2, 7, 26, 97, 362. . . .

ASSOCIATION FOR TEACHING AIDS IN MATHEMATICS

CONSTITUTION

1. The name of the Association shall be "THE ASSOCIATION FOR TEACHING AIDS IN MATHEMATICS."

2. The AIMS of the Association shall be to promote the study of the teaching of mathematics, and to improve the aids used therein, through week-end courses, exhibitions, demonstration lessons, and the publication of a bulletin.

3. MEMBERSHIP of the Association is open to all who are interested in these aims.

4. The SUBSCRIPTION shall be five shillings per annum, covering the calendar year, and shall be payable to the Treasurer from the beginning of the year. Subscriptions of new members joining after 1st October shall also cover membership for the following calendar year.

5. The COMMITTEE shall consist of a Chairman, Secretary, Treasurer and nine other members, and shall be responsible for all day to day management of the Association. It shall have power to co-opt additional members, to appoint assistants to Secretary and Treasurer, an Editorial staff for the bulletin, a Librarian, and to delegate responsibility to sub-committees set up to act for particular aspects of the work of the Association.

6. The Committee of the Association shall call an ANNUAL GENERAL MEETING once in the course of each calendar year, and at this Meeting the Committee for the subsequent year shall be elected. The Annual General Meeting shall also have power to fill the office of President for the ensuing year and to elect Vice-Presidents. Such Officers are to be ex-officio members of the Committee. Nominations, duly proposed and seconded and with the consent of the nominee, must reach the Secretary, in writing, at least 28 days before the date of the Annual General Meeting.

7. EXTRAORDINARY GENERAL MEETINGS may be called at any time, either by the Committee, or at the written request to the Secretary of at least twenty members of the Association, with reasonable time allowed for making the necessary arrangements.

8. A QUORUM for a General Meeting shall be 20 members.

9. The CONSTITUTION may be amended by a resolution passed by a simple majority at an Annual General Meeting. Notice of any such resolution must reach the Secretary in writing at least 28 days before the date of the Meeting.

ASSOCIATION FOR TEACHING AIDS IN MATHEMATICS

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The Annual Subscription to the Association is 5s for 1958.; cheques should be made payable to "A.T.A.M", and should be addressed to the Treasurer. This journal is supplied free to members.

The rate of subscription is to be raised to 10/- for 1959.